THE GAUSS–KRÜGER PROJECTION

R. E. Deakin[1], M. N. Hunter[2] and C. F. F. Karney[3]
[1] School of Mathematical and Geospatial Sciences, RMIT University
[2] Maribyrnong, VIC, Australia.
[3] Princeton, N.J., USA.
email: rod.deakin@rmit.edu.au

Original version: Presented at the Victorian Regional Survey Conference, Warrnambool, 10-12 September, 2010
This version with minor corrections: February 2022

ABSTRACT

The Gauss-Krüger projection appears to have two different forms. One is a set of equations capable of micrometre accuracy anywhere within 30° of a central meridian of longitude. The other is equations limited to millimetre accuracy within 3°– 6° of a central meridian. These latter equations are complicated but are in wide use in the geospatial community. The former equations are relatively simple (although they do contain hyperbolic functions) and are easily adapted to computers. But they are not in wide use. This paper will give some insight into both forms of the Gauss-Krüger projection.

INTRODUCTION

The Gauss-Krüger projection is a conformal mapping of a reference ellipsoid of the earth onto a plane where the equator and central meridian remain as straight lines and the scale along the central meridian is constant; all other meridians and parallels being complex curves (Figure 5).

The Gauss-Krüger is one of a family of transverse Mercator projections of which the spherical form was originally developed by Johann Heinrich Lambert (1728-1777). This projection is also called the Gauss-Lambert projection acknowledging the contribution of Carl Friedrich Gauss (1777–1855) to the development of the transverse Mercator projection. Snyder (1993) and Lee (1976) have excellent summaries of the history of development which we paraphrase below.

Gauss (c.1822) developed the ellipsoidal transverse Mercator as one example of his investigations in conformal transformations using complex algebra and used it for the survey of Hannover in the same decade. This projection had constant scale along the central meridian and was known as the Gauss conformal or Gauss' Hannover projection. Also (c. 1843) Gauss developed a 'double projection' consisting of a conformal mapping of the ellipsoid onto the sphere followed by a mapping from the sphere to the plane using the spherical transverse Mercator formula. This projection was adapted by Oskar Schreiber and used for the Prussian Land Survey of 1876-1923. It is also called the Gauss-Schreiber projection and scale along the central meridian is not constant. Gauss left few details of his original developments and Schreiber (1866, 1897) published an analysis of Gauss' methods, and Louis Krüger (1912) re-evaluated both Gauss' and Schreiber's work, hence the name Gauss-Krüger as a synonym for the transverse Mercator projection.

The aim of this paper is to give a detailed derivation of a set of equations that we call the Gauss-Krüger projection. These equations give micrometre accuracy anywhere within 30° of a central meridian; and at their heart are two important series linking conformal latitude ϕ' and rectifying latitude μ. We provide a development of these series noting our extensive use of the computer algebra systems MAPLE and Maxima in showing these series to high orders of n; unlike Krüger who only had patience. And without these computer tools it would be impossible to realize the potential of his series.

Krüger gave another set of equations that we would recognise as Thomas's or Redfearn's equations (Thomas 1952, Redfearn 1948). These other equations – also known as the Gauss-Krüger projection – are in wide use in the geospatial community; but they are complicated, and only accurate within a narrow band (3°– 6°) about a central meridian. We outline the development of these equations but do not give them explicitly, as we do not wish to promote their use. We also show that the use of these equations can lead to large errors in some circumstances.

This paper supports the work of Engsager & Poder (2007) who also use Krüger's series in their elegant algorithms for a highly accurate transverse Mercator projection but provide no derivation of the formulae. Also, one of the authors (Karney 2010) has a detailed analysis of the accuracy of our Gauss-Krüger projection equations and this paper may be regarded as background reading.

The preliminary sections set out below contain information that can be found in many geodesy and map projection texts and could probably be omitted but they are included here for completeness. As is the extensive Appendix that may be useful to the student following the development with pencil and paper at hand.

SOME PRELIMINARIES

The Gauss-Krüger projection is a mapping of a reference ellipsoid of the earth onto a plane and some definition of the ellipsoid and various associated constants are useful. We then give a limited introduction to differential geometry including definitions and formulae for the Gaussian fundamental quantities e, f and g, the differential distance ds and scale factors m, h and k.

Next, we define and give equations for the isometric latitude ψ, meridian distance M, quadrant length Q, the rectifying radius A and the rectifying latitude μ. This then provides the basic 'tools' to derive the conformal latitude ϕ' and show how the two important series linking ϕ' and μ are obtained.

The ellipsoid

In geodesy, the ellipsoid is a surface of revolution created by rotating an ellipse (whose semi-axes lengths are a and b and $a > b$) about its minor axis. The ellipsoid is the mathematical surface that idealizes the irregular shape of the earth and it has the following geometrical constants:

flattening $$f = \frac{a-b}{a} \tag{1}$$

eccentricity $$\varepsilon = \sqrt{\frac{a^2-b^2}{a^2}} \tag{2}$$

2nd eccentricity $$\varepsilon' = \sqrt{\frac{a^2-b^2}{b^2}} \tag{3}$$

3rd flattening $$n = \frac{a-b}{a+b} \tag{4}$$

polar radius $$c = \frac{a^2}{b} \tag{5}$$

These geometric constants are inter-related as follows

$$\frac{b}{a} = 1-f = \sqrt{1-\varepsilon^2} = \frac{1}{\sqrt{1+\varepsilon'^2}} = \frac{1-n}{1+n} = \frac{a}{c} \tag{6}$$

$$\varepsilon^2 = \frac{\varepsilon'^2}{1+\varepsilon'^2} = f\left(2-f\right) = \frac{4n}{\left(1+n\right)^2} \tag{7}$$

$$\varepsilon'^2 = \frac{\varepsilon^2}{1-\varepsilon^2} = \frac{f\left(2-f\right)}{\left(1-f\right)^2} = \frac{4n}{\left(1-n\right)^2} \tag{8}$$

$$n = \frac{f}{2-f} = \frac{1-\sqrt{1-\varepsilon^2}}{1+\sqrt{1-\varepsilon^2}} \tag{9}$$

From equation (7) an absolutely convergent series for ε^2 is

$$\varepsilon^2 = 4n - 8n^2 + 12n^3 - 16n^4 + 20n^5 - \cdots \tag{10}$$

since $0 < n < 1$.

The ellipsoid radii of curvature ρ (meridian plane) and υ (prime vertical plane) at a point whose latitude is ϕ are

$$\rho = \frac{a\left(1-\varepsilon^2\right)}{\left(1-\varepsilon^2\sin^2\phi\right)^{3/2}} = \frac{a\left(1-\varepsilon^2\right)}{W^3} = \frac{c}{V^3} \tag{11}$$

$$\upsilon = \frac{a}{\left(1-\varepsilon^2\sin^2\phi\right)^{1/2}} = \frac{a}{W} = \frac{c}{V} \tag{12}$$

where the latitude functions V and W are defined as

$$W^2 = 1-\varepsilon^2\sin^2\phi, \quad V^2 = 1+\varepsilon'^2\cos^2\phi \tag{13}$$

Some differential geometry: the differential rectangle and Gaussian fundamental quantities *e, f, g;* $\bar{E},\bar{F},\bar{G}$ and *E, F, G*

Curvilinear coordinates ϕ (latitude), λ (longitude) are used to define the location of points on the ellipsoid (the datum surface) and these points can also have x,y,z Cartesian coordinates where the positive z-axis is the rotational axis of the ellipsoid passing through the north pole, the x-y plane is the equatorial plane and the x-z plane is the Greenwich meridian plane. The positive x-axis passes through the intersection of the Greenwich meridian and equator and the positive y-axis is advanced 90° eastwards along the equator.

The curvilinear and Cartesian coordinates are related by

$$\begin{aligned} x &= x\left(\phi,\lambda\right) = \upsilon\cos\phi\cos\lambda \\ y &= y\left(\phi,\lambda\right) = \upsilon\cos\phi\sin\lambda \\ z &= z\left(\phi,\lambda\right) = \upsilon\left(1-\varepsilon^2\right)\sin\phi \end{aligned} \tag{14}$$

The differential arc-length ds of a curve on the ellipsoid is given by

$$\left(ds\right)^2 = \left(dx\right)^2 + \left(dy\right)^2 + \left(dz\right)^2 \tag{15}$$

And the total differentials are

$$\begin{aligned} dx &= \frac{\partial x}{\partial\phi}d\phi + \frac{\partial x}{\partial\lambda}d\lambda \\ dy &= \frac{\partial y}{\partial\phi}d\phi + \frac{\partial y}{\partial\lambda}d\lambda \\ dz &= \frac{\partial z}{\partial\phi}d\phi + \frac{\partial z}{\partial\lambda}d\lambda \end{aligned} \tag{16}$$

Substituting equations (16) into (15) gathering terms and simplifying gives

$$\left(ds\right)^2 = e\left(d\phi\right)^2 + 2f\,d\phi\,d\lambda + g\left(d\lambda\right)^2 \tag{17}$$

where the coefficients e, f and g are known as the Gaussian fundamental quantities and are given by

$$\begin{aligned} e &= \left(\frac{\partial x}{\partial\phi}\right)^2 + \left(\frac{\partial y}{\partial\phi}\right)^2 + \left(\frac{\partial z}{\partial\phi}\right)^2 \\ f &= \frac{\partial x}{\partial\phi}\frac{\partial x}{\partial\lambda} + \frac{\partial y}{\partial\phi}\frac{\partial y}{\partial\lambda} + \frac{\partial z}{\partial\phi}\frac{\partial z}{\partial\lambda} \\ g &= \left(\frac{\partial x}{\partial\lambda}\right)^2 + \left(\frac{\partial y}{\partial\lambda}\right)^2 + \left(\frac{\partial z}{\partial\lambda}\right)^2 \end{aligned} \tag{18}$$

For the ellipsoid

$$e = \rho^2, \quad f = 0, \quad g = \upsilon^2\cos^2\phi \tag{19}$$

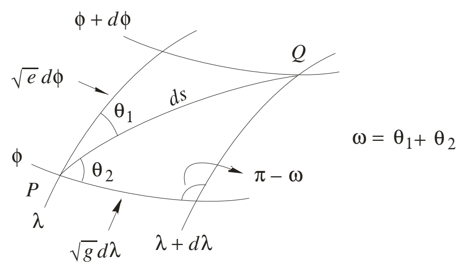

Figure 1. The differential rectangle

In Figure 1 the parametric curves ϕ and λ pass through *P* and the curves $\phi + d\phi$ and $\lambda + d\lambda$ pass through *Q,* and the length of the curve $PQ = ds$. The differential rectangle formed by the curves may be regarded as a plane figure whose opposite sides are parallel straight lines enclosing a differentially small area *da.*

The differential distances along the parametric curves are

$$\begin{aligned} ds_\lambda &= \sqrt{e}\, d\phi = \rho\, d\phi \\ ds_\phi &= \sqrt{g}\, d\lambda = \upsilon \cos\phi\, d\lambda \end{aligned} \tag{20}$$

and the angle ω between the parametric curves can be found from

$$\cos\omega = \frac{f}{\sqrt{eg}} \tag{21}$$

Thus, if the parametric curves on the surface intersect at right angles (i.e., they form an orthogonal system of curves) then $\omega = \frac{1}{2}\pi$ and $\cos\,\omega = 0$, and so from equation (21) $f = 0$. Conversely, if $f = 0$ the parametric curves form an orthogonal system.

Suppose there is another surface (projection surface) with Φ, Λ curvilinear coordinates and *X,Y,Z* Cartesian coordinates related by

$$X = X(\Phi, \Lambda), \quad Y = Y(\Phi, \Lambda), \quad Z = Z(\Phi, \Lambda) \tag{22}$$

Then, using equations (15) and (16) replacing *x,y,z* with *X,Y,Z* and ϕ, λ with Φ, Λ we have the differential distance *dS* on this surface defined by

$$(dS)^2 = \bar{E}(d\Phi)^2 + 2\bar{F}\, d\Phi\, d\Lambda + \bar{G}(d\Lambda)^2 \tag{23}$$

with the Gaussian fundamental quantities

$$\begin{aligned} \bar{E} &= \left(\frac{\partial X}{\partial \Phi}\right)^2 + \left(\frac{\partial Y}{\partial \Phi}\right)^2 + \left(\frac{\partial Z}{\partial \Phi}\right)^2 \\ \bar{F} &= \frac{\partial X}{\partial \Phi}\frac{\partial X}{\partial \Lambda} + \frac{\partial Y}{\partial \Phi}\frac{\partial Y}{\partial \Lambda} + \frac{\partial Z}{\partial \Phi}\frac{\partial Z}{\partial \Lambda} \\ \bar{G} &= \left(\frac{\partial X}{\partial \Lambda}\right)^2 + \left(\frac{\partial Y}{\partial \Lambda}\right)^2 + \left(\frac{\partial Z}{\partial \Lambda}\right)^2 \end{aligned} \tag{24}$$

Alternatively, the *X,Y,Z* Cartesian coordinates (projection surface) can be expressed as functions of the ϕ, λ curvilinear coordinates (datum surface) as

$$X = X(\phi, \lambda), \quad Y = Y(\phi, \lambda), \quad Z = Z(\phi, \lambda) \tag{25}$$

and

$$(dS)^2 = E(d\phi)^2 + 2F\, d\phi\, d\lambda + G(d\lambda)^2 \tag{26}$$

where

$$\begin{aligned} E &= \left(\frac{\partial X}{\partial \phi}\right)^2 + \left(\frac{\partial Y}{\partial \phi}\right)^2 + \left(\frac{\partial Z}{\partial \phi}\right)^2 \\ F &= \frac{\partial X}{\partial \phi}\frac{\partial X}{\partial \lambda} + \frac{\partial Y}{\partial \phi}\frac{\partial Y}{\partial \lambda} + \frac{\partial Z}{\partial \phi}\frac{\partial Z}{\partial \lambda} \\ G &= \left(\frac{\partial X}{\partial \lambda}\right)^2 + \left(\frac{\partial Y}{\partial \lambda}\right)^2 + \left(\frac{\partial Z}{\partial \lambda}\right)^2 \end{aligned} \tag{27}$$

Scale factors *m, h* and *k*

The scale factor *m* is defined as the ratio of differential distances *dS* (projection surface) and *ds* (datum surface) and is usually given as a squared value

$$\begin{aligned} m^2 = \frac{(dS)^2}{(ds)^2} &= \frac{\bar{E}(d\Phi)^2 + 2\bar{F}\, d\Phi\, d\Lambda + \bar{G}(d\Lambda)^2}{e(d\phi)^2 + 2f\, d\phi\, d\lambda + g(d\lambda)^2} \\ &= \frac{E(d\phi)^2 + 2F\, d\phi\, d\lambda + G(d\lambda)^2}{e(d\phi)^2 + 2f\, d\phi\, d\lambda + g(d\lambda)^2} \end{aligned} \tag{28}$$

When $\dfrac{\bar{E}}{e} = \dfrac{\bar{G}}{g}$ and $\bar{F} = 0$ or $\dfrac{E}{e} = \dfrac{G}{g}$ and $F = 0$ the scale factor *m* is the same in every direction and such projections are known as conformal. For the ellipsoid (datum surface) where the parametric curves ϕ, λ are an orthogonal system and $f = 0$, this scale condition for conformal projection of the ellipsoid is often expressed as

$$h = k \quad \text{and} \quad F = \bar{F} = 0 \tag{29}$$

h is the scale factor along the meridian and *k* is the scale factor along the parallel of latitude. Using equations (28)

$$h = \frac{\sqrt{\bar{E}}\, d\Phi}{\sqrt{e}\, d\phi} = \frac{\sqrt{E}}{\sqrt{e}} \quad \text{and} \quad k = \frac{\sqrt{\bar{G}}\, d\Lambda}{\sqrt{g}\, d\lambda} = \frac{\sqrt{G}}{\sqrt{g}} \tag{30}$$

Isometric latitude ψ

According to the SOED (1993) isometric means: "of equal measure or dimension" and we may think of isometric parameters ψ (isometric latitude) and $\omega = \lambda - \lambda_0$ (longitude difference) in the following way. Imagine you are standing on the earth at the equator and you measure a metre north and a metre east; both of these equal lengths would represent almost equal angular changes in latitude $d\phi$ and longitude $d\lambda$. Now imagine you are close to the north pole; a metre in the north direction will represent (almost) the same angular change $d\phi$ as it did at the equator. But a metre in the east direction would represent a much greater change in longitude, i.e., equal north and east linear measures near the pole do not correspond to equal angular measures.

What is required is isometric latitude ψ, a variable angular measure along a meridian that is defined by considering the differential rectangle of Figure 1 and equations (17) and (19) giving

$$\begin{aligned} ds^2 &= \left(\sqrt{e}\,d\phi\right)^2 + \left(\sqrt{g}\,d\lambda\right)^2 \\ &= \rho^2\left(d\phi\right)^2 + \upsilon^2\cos^2\phi\left(d\lambda\right)^2 \\ &= \upsilon^2\cos^2\phi\left\{\left(\frac{\rho\,d\phi}{\upsilon\cos\phi}\right)^2 + \left(d\lambda\right)^2\right\} \\ &= \upsilon^2\cos^2\phi\left\{\left(d\psi\right)^2 + \left(d\lambda\right)^2\right\} \end{aligned} \tag{31}$$

where isometric latitude ψ is defined by the relationship

$$d\psi = \frac{\rho}{\upsilon\cos\phi}\,d\phi \tag{32}$$

Integrating this gives (Deakin & Hunter 2010b)

$$\psi = \ln\left\{\tan\left(\tfrac{1}{4}\pi + \tfrac{1}{2}\phi\right)\left(\frac{1-\varepsilon\sin\phi}{1+\varepsilon\sin\phi}\right)^{\frac{1}{2}\varepsilon}\right\} \tag{33}$$

Note that if the reference surface for the earth is a sphere of radius *R;* then $\rho = \upsilon = R$, $\varepsilon = 0$ and the isometric latitude is

$$\psi = \ln\tan\left(\tfrac{1}{4}\pi + \tfrac{1}{2}\phi\right) \tag{34}$$

Meridian distance *M*

Meridian distance *M* is defined as the arc of the meridian from the equator to the point of latitude ϕ

$$M = \int_0^{\phi}\rho\,d\phi = \int_0^{\phi}\frac{a\left(1-\varepsilon^2\right)}{W^3}\,d\phi = \int_0^{\phi}\frac{c}{V^3}\,d\phi \tag{35}$$

This is an elliptic integral that cannot be expressed in terms of elementary functions; instead, the integrand is expanded by use of the binomial series then the integral is evaluated by term-by-term integration. The usual form of the series formula for *M* is a function of ϕ and powers of ε^2; but the German geodesist F.R. Helmert (1880) gave a formula for meridian distance as a function of latitude ϕ and powers of the ellipsoid constant *n* that required fewer terms for the same accuracy than meridian distance formula involving powers of ε^2. Using Helmert's method of development (Deakin & Hunter 2010a) a formula for *M* can be written as

$$M = a\left(1-n\right)\left(1-n^2\right)\left\{\begin{aligned} &b_0\phi - b_2\sin 2\phi + b_4\sin 4\phi \\ &-b_6\sin 6\phi + b_8\sin 8\phi \\ &-b_{10}\sin 10\phi + \cdots \end{aligned}\right\} \tag{36}$$

where the coefficients $\{b_n\}$ have the following form

$$\begin{aligned} b_0 &= 1 + \left(\frac{3}{2}\right)^2 n^2 + \left(\frac{3\cdot 5}{2\cdot 4}\right)^2 n^4 + \left(\frac{3\cdot 5\cdot 7}{2\cdot 4\cdot 6}\right)^2 n^6 \\ &\quad + \left(\frac{3\cdot 5\cdot 7\cdot 9}{2\cdot 4\cdot 6\cdot 8}\right)^2 n^8 + \cdots \\ b_2 &= \frac{2}{2}\left(\begin{aligned} &\frac{3}{2}n + \left(\frac{3}{2}\right)\left(\frac{3\cdot 5}{2\cdot 4}\right)n^3 + \left(\frac{3\cdot 5}{2\cdot 4}\right)\left(\frac{3\cdot 5\cdot 7}{2\cdot 4\cdot 6}\right)n^5 \\ &+\left(\frac{3\cdot 5\cdot 7}{2\cdot 4\cdot 6}\right)\left(\frac{3\cdot 5\cdot 7\cdot 9}{2\cdot 4\cdot 6\cdot 8}\right)n^7 + \cdots \end{aligned}\right) \\ b_4 &= \frac{2}{4}\left(\begin{aligned} &\left(\frac{3\cdot 5}{2\cdot 4}\right)n^2 + \left(\frac{3}{2}\right)\left(\frac{3\cdot 5\cdot 7}{2\cdot 4\cdot 6}\right)n^4 \\ &+\left(\frac{3\cdot 5}{2\cdot 4}\right)\left(\frac{3\cdot 5\cdot 7\cdot 9}{2\cdot 4\cdot 6\cdot 8}\right)n^6 + \cdots \end{aligned}\right) \\ b_6 &= \frac{2}{6}\left(\begin{aligned} &\left(\frac{3\cdot 5\cdot 7}{2\cdot 4\cdot 6}\right)n^3 + \left(\frac{3}{2}\right)\left(\frac{3\cdot 5\cdot 7\cdot 9}{2\cdot 4\cdot 6\cdot 8}\right)n^5 \\ &+\left(\frac{3\cdot 5}{2\cdot 4}\right)\left(\frac{3\cdot 5\cdot 7\cdot 9\cdot 11}{2\cdot 4\cdot 6\cdot 8\cdot 10}\right)n^7 + \cdots \end{aligned}\right) \\ b_8 &= \frac{2}{8}\left(\begin{aligned} &\left(\frac{3\cdot 5\cdot 7\cdot 9}{2\cdot 4\cdot 6\cdot 8}\right)n^4 + \left(\frac{3}{2}\right)\left(\frac{3\cdot 5\cdot 7\cdot 9\cdot 11}{2\cdot 4\cdot 6\cdot 8\cdot 10}\right)n^6 \\ &+\left(\frac{3\cdot 5}{2\cdot 4}\right)\left(\frac{3\cdot 5\cdot 7\cdot 9\cdot 11\cdot 13}{2\cdot 4\cdot 6\cdot 8\cdot 10\cdot 12}\right)n^8 + \cdots \end{aligned}\right) \\ b_{10} &= \frac{2}{10}\left(\begin{aligned} &\left(\frac{3\cdot 5\cdot 7\cdot 9\cdot 11}{2\cdot 4\cdot 6\cdot 8\cdot 10}\right)n^5 \\ &+\left(\frac{3}{2}\right)\left(\frac{3\cdot 5\cdot 7\cdot 9\cdot 11\cdot 13}{2\cdot 4\cdot 6\cdot 8\cdot 10\cdot 12}\right)n^7 + \cdots \end{aligned}\right) \end{aligned}$$

Noting that

$$\begin{aligned} \left(1-n\right)\left(1-n^2\right) &= \frac{1+n}{1+n}\left(1-n\right)\left(1-n^2\right) \\ &= \frac{\left(1-n^2\right)\left(1-n^2\right)}{1+n} = \frac{\left(1-n^2\right)^2}{1+n} \end{aligned}$$

then multiplying the coefficients $b_0, b_2, b_4, \ldots$ in equation (36) by $\left(1-n^2\right)^2$ gives

$$M = \frac{a}{1+n}\left\{\begin{aligned} &c_0\phi + c_2\sin 2\phi + c_4\sin 4\phi \\ &+c_6\sin 6\phi + c_8\sin 8\phi \\ &+c_{10}\sin 10\phi + c_{12}\sin 12\phi \\ &+c_{14}\sin 14\phi + c_{16}\sin 16\phi + \cdots \end{aligned}\right\} \tag{37}$$

where the coefficients $\{c_n\}$ are to order n^8 as follows

$$
\begin{aligned}
c_0 &= 1+\frac{1}{4}n^2+\frac{1}{64}n^4+\frac{1}{256}n^6+\frac{25}{16384}n^8\cdots\\
c_2 &= -\frac{3}{2}n+\frac{3}{16}n^3+\frac{3}{128}n^5+\frac{15}{2048}n^7+\cdots\\
c_4 &= \frac{15}{16}n^2-\frac{15}{64}n^4-\frac{75}{2048}n^6-\frac{105}{8192}n^8-\cdots\\
c_6 &= -\frac{35}{48}n^3+\frac{175}{768}n^5+\frac{245}{6144}n^7+\cdots\\
c_8 &= \frac{315}{512}n^4-\frac{441}{2048}n^6-\frac{1323}{32768}n^8-\cdots\\
c_{10} &= -\frac{693}{1280}n^5+\frac{2079}{10240}n^7+\cdots\\
c_{12} &= \frac{1001}{2048}n^6-\frac{1573}{8192}n^8-\cdots\\
c_{14} &= -\frac{6435}{14336}n^7+\cdots\\
c_{16} &= \frac{109395}{262144}n^8-\cdots
\end{aligned} \tag{38}
$$

[This is Krüger's equation for X shown in §5, p.12, extended to order n^8]

Quadrant length *Q*

The quadrant length of the ellipsoid Q is the length of the meridian arc from the equator to the pole and is obtained from equation (37) by setting $\phi = \frac{1}{2}\pi$, and noting that $\sin 2\phi, \sin 4\phi, \sin 6\phi, \ldots$ all equal zero, giving

$$Q = \frac{a\pi}{2(1+n)}c_0 \tag{39}$$

[This is Krüger's equation for $\mathfrak{M}$ shown in §5, p.12.]

Rectifying radius *A*

Dividing the quadrant length Q by $\frac{1}{2}\pi$ gives the rectifying radius A, which is the radius of a circle having the same circumference as the meridian ellipse and A to order n^8 is

$$A = \frac{a}{1+n}\left\{1+\frac{1}{4}n^2+\frac{1}{64}n^4+\frac{1}{256}n^6+\frac{25}{16384}n^8+\cdots\right\} \tag{40}$$

Rectifying latitude μ

The rectifying latitude μ is defined in the following way (Adams 1921):

> “If a sphere is determined such that the length of a great circle upon it is equal in length to a meridian upon the earth, we may calculate the latitudes upon this sphere such that the arcs of the meridian upon it are equal to the corresponding arcs of the meridian upon the earth.”

If μ denotes this latitude on the sphere of radius R then meridian distance M is given by

$$M = R\mu \tag{41}$$

and since $\mu = \frac{1}{2}\pi$ when $M = Q$ then $R = A$ and the rectifying latitude μ is defined as

$$\mu = \frac{M}{A} \tag{42}$$

An expression for μ as a function of ϕ and powers of n is obtained by dividing equation (40) into equation (37) giving to order n^4

$$
\begin{aligned}
\mu = \phi &+ d_2\sin 2\phi + d_4\sin 4\phi + d_6\sin 6\phi\\
&+ d_8\sin 8\phi + \cdots
\end{aligned} \tag{43}
$$

where the coefficients $\{d_n\}$ are

$$
\begin{aligned}
d_2 &= -\frac{3}{2}n+\frac{9}{16}n^3-\cdots\\
d_4 &= \frac{15}{16}n^2-\frac{15}{32}n^4+\cdots\\
d_6 &= -\frac{35}{48}n^3+\cdots\\
d_8 &= \frac{315}{512}n^4-\cdots
\end{aligned} \tag{44}
$$

[This is Krüger's equation (6), §5, p. 12.]

An expression for ϕ as a function of μ and powers of n is obtained by reversion of a series using Lagrange's theorem (see Appendix) and to order n^4

$$
\begin{aligned}
\phi = \mu &+ D_2\sin 2\mu + D_4\sin 4\mu + D_6\sin 6\mu\\
&+ D_8\sin 8\mu + \cdots
\end{aligned} \tag{45}
$$

where the coefficients $\{D_n\}$ are

$$
\begin{aligned}
D_2 &= \frac{3}{2}n-\frac{27}{32}n^3+\cdots\\
D_4 &= \frac{21}{16}n^2-\frac{55}{32}n^4+\cdots\\
D_6 &= \frac{151}{96}n^3-\cdots\\
D_8 &= \frac{1097}{512}n^4-\cdots
\end{aligned} \tag{46}
$$

[This is Krüger's equation (7), §5, p. 13.]

Conformal latitude ϕ'

Suppose we have a sphere of radius a with curvilinear coordinates ϕ', λ' (meridians and parallels) and X,Y,Z Cartesian coordinates related by

$$
\begin{aligned}
X &= X(\phi',\lambda') = a\cos\phi'\cos\lambda'\\
Y &= Y(\phi',\lambda') = a\cos\phi'\sin\lambda'\\
Z &= Z(\phi',\lambda') = a\sin\phi'
\end{aligned} \tag{47}
$$

Substituting equations (47) into equations (24), replacing Φ, Λ with ϕ', λ' gives the Gaussian Fundamental Quantities for the sphere as

$$\bar{E} = a^2, \quad \bar{F} = 0, \quad \bar{G} = a^2\cos^2\phi' \tag{48}$$

The conformal projection of the ellipsoid (datum surface) onto the sphere (projection surface) is obtained by enforcing the condition $h = k$; and with equations (19), (30) and (48), replacing Φ, Λ with ϕ', λ' where appropriate, we have,

$$\frac{a\,d\phi'}{\rho\,d\phi} = \frac{a\cos\phi'\,d\lambda'}{\upsilon\cos\phi\,d\lambda} \tag{49}$$

This differential equation can be simplified by enforcing the condition that the scale factor along the equator be unity, so that

$$\frac{a\cos\phi_0'\,d\lambda'}{\upsilon_0\cos\phi_0\,d\lambda} = 1$$

and since $\phi_0 = \phi_0' = 0$ then $\cos\phi_0 = \cos\phi_0' = 1$, $\upsilon_0 = a$ and $d\lambda = d\lambda'$. Substituting this result into equation (49) gives

$$\frac{d\phi'}{\cos\phi'} = \frac{\rho\,d\phi}{\upsilon\cos\phi} \tag{50}$$

Integrating both sides gives

$$\ln\tan\left(\tfrac{1}{4}\pi + \tfrac{1}{2}\phi'\right) = \ln\left\{\tan\left(\tfrac{1}{4}\pi + \tfrac{1}{2}\phi\right)\left(\frac{1-\varepsilon\sin\phi}{1+\varepsilon\sin\phi}\right)^{\frac{1}{2}\varepsilon}\right\} \tag{51}$$

The conformal longitude λ' is obtained from the differential relationship $d\lambda = d\lambda'$ which is a consequence of the scale factor along the equator being unity, and longitude on the ellipsoid is identical to longitude on the conformal sphere, which makes

$$\lambda' = \lambda \tag{52}$$

Series involving conformal latitude and rectifying latitude

Two series that are crucial to the Gauss-Krüger projection will be developed in this section; they are (i) a series for conformal latitude ϕ' as a function of the rectifying latitude μ, and (ii) a series for μ as a function of ϕ'. The method of development is not the same as employed by Krüger, but does give some insight into the power and use of Taylor's theorem (see Appendix). Also, we should remember that Krüger had only pencil, paper and perseverance whilst we have the power of computer algebra systems.

A series for conformal latitude ϕ' as a function of latitude ϕ may be developed using a method given by Yang *et al.* (2000).

The right-hand-side of equation (51) is the isometric latitude ψ and we write

$$\exp(\psi) = \tan\left(\tfrac{1}{4}\pi + \tfrac{1}{2}\phi'\right) \tag{53}$$

where $\exp(\psi) = \mathrm{e}^{\psi}$ is the exponential function.

Solving equation (53) gives the conformal latitude

$$\phi' = 2\tan^{-1}\left(\exp(\psi)\right) - \tfrac{1}{2}\pi \tag{54}$$

Let $F(\phi,\varepsilon) = \tan^{-1}\left(\exp(\psi)\right)$ so that equation (54) becomes

$$\phi' = 2F(\phi,\varepsilon) - \tfrac{1}{2}\pi \tag{55}$$

and

$$F(\phi,\varepsilon) = \tan^{-1}\left\{\tan\left(\tfrac{1}{4}\pi + \tfrac{1}{2}\phi\right)\left(\frac{1-\varepsilon\sin\phi}{1+\varepsilon\sin\phi}\right)^{\frac{1}{2}\varepsilon}\right\}$$

Now since the eccentricity ε satisfies $0 < \varepsilon < 1$, $F(\phi,\varepsilon)$ may be expanded into a power series of ε about $\varepsilon = 0$ using Taylor's theorem [see Appendix, equation (139)]

$$\begin{aligned} F(\phi,\varepsilon) = F(\phi,\varepsilon)\big|_{\varepsilon=0} &+ \varepsilon\left[\frac{\partial}{\partial\varepsilon}F(\phi,\varepsilon)\right]_{\varepsilon=0} \\ &+ \frac{\varepsilon^2}{2!}\left[\frac{\partial^2}{\partial\varepsilon^2}F(\phi,\varepsilon)\right]_{\varepsilon=0} \\ &+ \frac{\varepsilon^3}{3!}\left[\frac{\partial^3}{\partial\varepsilon^3}F(\phi,\varepsilon)\right]_{\varepsilon=0} \\ &+ \cdots \end{aligned} \tag{56}$$

All odd-order partial derivatives evaluated at $\varepsilon = 0$ are zero and the even-order partial derivatives evaluated at $\varepsilon = 0$ are

$$\begin{aligned} \left[\frac{\partial^2}{\partial\varepsilon^2}F(\phi,\varepsilon)\right]_{\varepsilon=0} &= -\frac{1}{2}\sin 2\phi \\ \left[\frac{\partial^4}{\partial\varepsilon^4}F(\phi,\varepsilon)\right]_{\varepsilon=0} &= -\frac{5}{2}\sin 2\phi + \frac{5}{4}\sin 4\phi \\ \left[\frac{\partial^6}{\partial\varepsilon^6}F(\phi,\varepsilon)\right]_{\varepsilon=0} &= \left\{\begin{array}{l} -\frac{135}{4}\sin 2\phi + \frac{63}{2}\sin 4\phi \\ -\frac{39}{4}\sin 6\phi \end{array}\right\} \\ \left[\frac{\partial^8}{\partial\varepsilon^8}F(\phi,\varepsilon)\right]_{\varepsilon=0} &= \left\{\begin{array}{l} -\frac{1967}{2}\sin 2\phi + \frac{4879}{4}\sin 4\phi \\ -\frac{1383}{2}\sin 6\phi + \frac{1237}{8}\sin 8\phi \end{array}\right\} \\ &\cdots \end{aligned}$$

[We note that the last of these derivatives is incorrectly shown in Yang *et al.* (2000, p. 80).]

Substituting these partial derivatives with $F(\phi,\varepsilon)\big|_{\varepsilon=0} = \tfrac{1}{4}\pi + \tfrac{1}{2}\phi$ into equation (56), and re-arranging them into equation (55) gives

$$\phi' = \phi + \begin{pmatrix} -\frac{1}{2}\varepsilon^2 - \frac{5}{24}\varepsilon^4 - \frac{3}{32}\varepsilon^6 \\ -\frac{281}{5760}\varepsilon^8 - \cdots \end{pmatrix} \sin 2\phi$$
$$+\left(\frac{5}{48}\varepsilon^4 + \frac{7}{80}\varepsilon^6 + \frac{697}{11520}\varepsilon^8 + \cdots\right)\sin 4\phi$$
$$+\left(-\frac{13}{480}\varepsilon^6 - \frac{461}{13440}\varepsilon^8 - \cdots\right)\sin 6\phi$$
$$+\left(\frac{1237}{161280}\varepsilon^8 + \cdots\right)\sin 8\phi$$
$$+\cdots \qquad (57)$$

[Note that equation (57) is incorrectly shown in Yang *et al.* (2000, eqn (3.5.8), p. 80) due to the error noted previously.]

Extending the process outlined above to higher even-powers of ε and higher even-multiples of ϕ; and then using the series (10) to replace $\varepsilon^2, \varepsilon^4, \varepsilon^6, \ldots$ with powers of *n* gives a series for conformal latitude ϕ' as a function of latitude ϕ to order n^4 as

$$\phi' = \phi + g_2 \sin 2\phi + g_4 \sin 4\phi + g_6 \sin 6\phi + g_8 \sin 8\phi + \cdots \qquad (58)$$

where the coefficients $\{g_n\}$ are

$$\begin{aligned} g_2 &= -2n + \frac{2}{3}n^2 + \frac{4}{3}n^3 - \frac{82}{45}n^4 + \cdots \\ g_4 &= \frac{5}{3}n^2 - \frac{16}{15}n^3 - \frac{13}{9}n^4 + \cdots \\ g_6 &= -\frac{26}{15}n^3 + \frac{34}{21}n^4 + \cdots \\ g_8 &= \frac{1237}{630}n^4 - \cdots \end{aligned} \qquad (59)$$

[This is Krüger's equation (8), §5, p. 14.]

A series for conformal latitude ϕ' as a function of rectifying latitude μ can be obtained by using a method set out in Krüger (1912, p.14) that involves Taylor's theorem (see Appendix); where

$$\sin 2\phi = \sin 2\mu + (2\phi - 2\mu)\cos 2\mu - \frac{(2\phi - 2\mu)^2}{2!}\sin 2\mu - \frac{(2\phi - 2\mu)^3}{3!}\cos 2\mu + \frac{(2\phi - 2\mu)^4}{4!}\sin 2\mu + \cdots$$

Replacing 2ϕ and 2μ with 4ϕ and 4μ; 6ϕ and 6μ; etc.; and with expressions for $2\phi - 2\mu$, $4\phi - 4\mu$, etc. from equation (45) we can obtain expressions for $\sin 2\phi, \sin 4\phi, \sin 6\phi,$ etc. as functions of *n* and $\sin 2\mu, \sin 4\mu, \sin 6\mu,$ etc. Substituting these expressions into equation (58) and simplifying gives a series for conformal latitude ϕ' as a function of rectifying latitude μ to order n^8

$$\begin{aligned} \phi' = \mu &+ \beta_2 \sin 2\mu + \beta_4 \sin 4\mu + \beta_6 \sin 6\mu \\ &+ \beta_8 \sin 8\mu + \beta_{10} \sin 10\mu + \beta_{12} \sin 12\mu \\ &+ \beta_{14} \sin 14\mu + \beta_{16} \sin 16\mu + \cdots \end{aligned} \qquad (60)$$

where the coefficients $\{\beta_n\}$ are

$$\begin{aligned} \beta_2 &= -\frac{1}{2}n + \frac{2}{3}n^2 - \frac{37}{96}n^3 + \frac{1}{360}n^4 + \frac{81}{512}n^5 \\ &\quad - \frac{96199}{604800}n^6 + \frac{5406467}{38707200}n^7 \\ &\quad - \frac{7944359}{67737600}n^8 + \cdots \\ \beta_4 &= -\frac{1}{48}n^2 - \frac{1}{15}n^3 + \frac{437}{1440}n^4 - \frac{46}{105}n^5 \\ &\quad + \frac{1118711}{3870720}n^6 - \frac{51841}{1209600}n^7 \\ &\quad - \frac{24749483}{348364800}n^8 + \cdots \\ \beta_6 &= -\frac{17}{480}n^3 + \frac{37}{840}n^4 + \frac{209}{4480}n^5 - \frac{5569}{90720}n^6 \\ &\quad - \frac{9261899}{58060800}n^7 + \frac{6457463}{17740800}n^8 - \cdots \\ \beta_8 &= -\frac{4397}{161280}n^4 + \frac{11}{504}n^5 + \frac{830251}{7257600}n^6 \\ &\quad - \frac{466511}{2494800}n^7 - \frac{324154477}{7664025600}n^8 + \cdots \\ \beta_{10} &= -\frac{4583}{161280}n^5 + \frac{108847}{3991680}n^6 + \frac{8005831}{63866880}n^7 \\ &\quad - \frac{22894433}{124540416}n^8 - \cdots \\ \beta_{12} &= -\frac{20648693}{638668800}n^6 + \frac{16363163}{518918400}n^7 \\ &\quad + \frac{2204645983}{12915302400}n^8 - \cdots \\ \beta_{14} &= -\frac{219941297}{5535129600}n^7 + \frac{497323811}{12454041600}n^8 + \cdots \\ \beta_{16} &= -\frac{191773887257}{3719607091200}n^8 + \cdots \end{aligned} \qquad (61)$$

[This is Krüger's equation (10), §5, p. 14 extended to order n^8]

An expression for rectifying latitude μ as a function of conformal latitude ϕ' and powers of *n* is obtained by reversion of a series using Lagrange's theorem (see Appendix).

$$\begin{aligned} \mu = \phi' &+ \alpha_2 \sin 2\phi' + \alpha_4 \sin 4\phi' + \alpha_6 \sin 6\phi' + \alpha_8 \sin 8\phi' \\ &+ \alpha_{10} \sin 10\phi' + \alpha_{12} \sin 12\phi' + \alpha_{14} \sin 14\phi' \\ &+ \alpha_{16} \sin 16\phi' + \cdots \end{aligned} \qquad (62)$$

where the coefficients $\{\alpha_n\}$ are

$$
\begin{aligned}
\alpha_2 &= \frac{1}{2}n - \frac{2}{3}n^2 + \frac{5}{16}n^3 + \frac{41}{180}n^4 - \frac{127}{288}n^5 + \frac{7891}{37800}n^6 \\
&\quad + \frac{72161}{387072}n^7 - \frac{18975107}{50803200}n^8 + \cdots \\
\alpha_4 &= \frac{13}{48}n^2 - \frac{3}{5}n^3 + \frac{557}{1440}n^4 + \frac{281}{630}n^5 - \frac{1983433}{1935360}n^6 \\
&\quad + \frac{13769}{28800}n^7 + \frac{148003883}{174182400}n^8 - \cdots \\
\alpha_6 &= \frac{61}{240}n^3 - \frac{103}{140}n^4 + \frac{15061}{26880}n^5 + \frac{167603}{181440}n^6 \\
&\quad - \frac{67102379}{29030400}n^7 + \frac{79682431}{79833600}n^8 + \cdots \\
\alpha_8 &= \frac{49561}{161280}n^4 - \frac{179}{168}n^5 + \frac{6601661}{7257600}n^6 + \frac{97445}{49896}n^7 \\
&\quad - \frac{40176129013}{7664025600}n^8 + \cdots \\
\alpha_{10} &= \frac{34729}{80640}n^5 - \frac{3418889}{1995840}n^6 + \frac{14644087}{9123840}n^7 \\
&\quad + \frac{2605413599}{622702080}n^8 + \cdots \\
\alpha_{12} &= \frac{212378941}{319334400}n^6 - \frac{30705481}{10378368}n^7 \\
&\quad + \frac{175214326799}{58118860800}n^8 + \cdots \qquad (63) \\
\alpha_{14} &= \frac{1522256789}{1383782400}n^7 - \frac{16759934899}{3113510400}n^8 + \cdots \\
\alpha_{16} &= \frac{1424729850961}{743921418240}n^8 - \cdots
\end{aligned}
$$

[This is Krüger's equation (11), §5, p. 14 extended to order n^8]

THE GAUSS-KRÜGER PROJECTION

The Gauss-Krüger projection is the result of a triple-mapping in two parts (Bugayevskiy & Snyder 1995). The first part is a conformal mapping of the ellipsoid to a sphere (the conformal sphere of radius *a*) followed by a conformal mapping of this sphere to the plane using the spherical transverse Mercator projection equations with spherical latitude ϕ replaced by conformal latitude ϕ'.

This two-step process is also known as the Gauss-Schreiber projection (Snyder 1993) and the scale along the central meridian is not constant. [Note that the Gauss-Schreiber projection commonly uses a conformal sphere of radius $R = \sqrt{\rho_0 \upsilon_0}$ where ρ_0 and υ_0 are evaluated at a central latitude for the region of interest.] The second part is the conformal mapping from the Gauss-Schreiber to the Gauss-Krüger projection where the scale factor along the central meridian is made constant.

To understand this process we first discuss the spherical Mercator and transverse Mercator projections. We then give the equations for the Gauss-Schreiber projection and show that the scale factor along the central meridian of this projection is not constant. Finally, using complex functions and principles of conformal mapping developed by Gauss, we show the conformal mapping from the Gauss-Schreiber projection to the Gauss-Krüger projection.

Having established the 'forward' mapping $\phi, \lambda \to X, Y$ from the ellipsoid to the plane – via the conformal sphere and the Gauss-Schreiber projection – we show how the 'inverse' mapping $X, Y \to \phi, \lambda$ from the plane to the ellipsoid is achieved.

In addition to the equations for the forward and inverse mappings we derive equations for scale factor *m* and grid convergence γ.

Mercator projection of the sphere

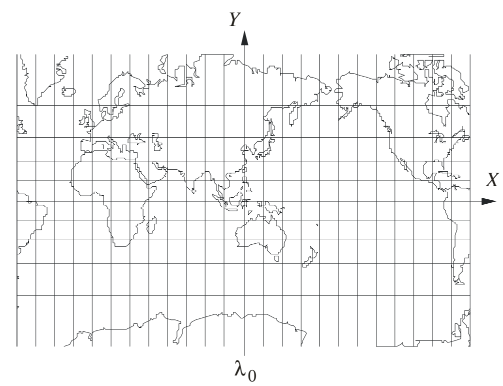

Figure 2 Mercator projection
graticule interval 15°, central meridian $\lambda_0 = 120^\circ \mathrm{E}$

The Mercator projection of the sphere is a conformal projection with the datum surface a sphere of radius *R* with curvilinear coordinates ϕ, λ and Gaussian fundamental quantities

$$e = R^2, \quad f = 0, \quad g = R^2 \cos^2 \phi \qquad (64)$$

The projection surface is a plane with *X,Y* Cartesian coordinates and $X = X(\lambda)$ and $Y = Y(\phi)$ and Gaussian fundamental quantities

$$E = \left(\frac{\partial Y}{\partial \phi}\right)^2, \quad F = 0, \quad G = \left(\frac{\partial X}{\partial \lambda}\right)^2 \qquad (65)$$

Enforcing the scale condition $h = k$ and using equations (30), (64) and (65) gives the differential equation

$$\frac{dY}{d\phi} = \frac{1}{\cos\phi}\frac{dX}{d\lambda} \qquad (66)$$

This equation can be simplified by enforcing the condition that the scale factor along the equator be unity giving

$$dX = R\, d\lambda \quad \text{and} \quad dY = \frac{1}{\cos\phi} d\phi$$

Integrating and evaluating constants of integration gives the well known equations for Mercator's projection of the sphere

$$
\begin{aligned}
X &= R(\lambda - \lambda_0) = R\,\omega \\
Y &= R \ln \tan\left(\tfrac{1}{4}\pi + \tfrac{1}{2}\phi\right)
\end{aligned} \tag{67}
$$

where λ_0 is the longitude of the central meridian and $\omega = \lambda - \lambda_0$.

An alternative set of equations for Mercator's projection may be derived as follows by using the half-angle formula $\tan\frac{x}{2} = \sqrt{\frac{1-\cos x}{1+\cos x}}$ and writing $\tan\left(\tfrac{1}{4}\pi + \tfrac{1}{2}\phi\right) = \sqrt{\frac{1-\cos\left(\tfrac{1}{2}\pi+\phi\right)}{1+\cos\left(\tfrac{1}{2}\pi+\phi\right)}} = \sqrt{\frac{1+\sin\phi}{1-\sin\phi}}$. Using this result in equation (67) gives

$$
\begin{aligned}
X &= R(\lambda - \lambda_0) = R\,\omega \\
Y &= R \ln \sqrt{\frac{1+\sin\phi}{1-\sin\phi}} = \frac{R}{2}\ln\left(\frac{1+\sin\phi}{1-\sin\phi}\right)
\end{aligned} \tag{68}
$$

Transverse Mercator projection of the sphere (Gauss-Lambert projection)

The equations for the transverse Mercator projection of the sphere (also known as the Gauss-Lambert projection) can be derived by considering the schematic view of the sphere in Figure 3 that shows *P* having curvilinear coordinates ϕ, λ that are angular quantities measured along great circles (meridian and equator).

Now consider the great circle *NBS* (the oblique equator) with a pole *A* that lies on the equator and great circles through *A*, one of which passes through *P* making an angle θ with the equator and intersecting the oblique equator at *C*.

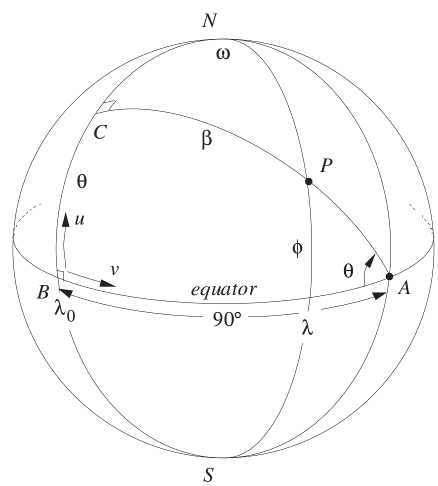

Figure 3 Oblique pole *A* on equator

β and θ are oblique latitude and oblique longitude respectively and equations linking β,θ and ϕ,λ can be obtained from spherical trigonometry and the right-angled spherical triangle *CNP* having sides β, $\tfrac{1}{2}\pi - \theta$, $\tfrac{1}{2}\pi - \phi$ and an angle at *N* of $\omega = \lambda - \lambda_0$

$$\sin\beta = \cos\phi \sin\omega \tag{69}$$

$$\tan\theta = \frac{\tan\phi}{\cos\omega} \tag{70}$$

Squaring both sides of equation (69) and using the trigonometric identity $\sin^2 x + \cos^2 x = 1$ gives, after some algebra

$$\cos\beta = \cos\phi\sqrt{\tan^2\phi + \cos^2\omega} \tag{71}$$

From equations (71) and (69)

$$\tan\beta = \frac{\sin\omega}{\sqrt{\tan^2\phi + \cos^2\omega}} \tag{72}$$

Replacing ϕ with β and ω with θ in equations (68); then using equations (69), (70) and (144) give the equations for the transverse Mercator projection of the sphere (Lauf 1983, Snyder 1987)

$$
\begin{aligned}
u &= R\theta \\
&= R\tan^{-1}\left(\frac{\tan\phi}{\cos\omega}\right) \\
v &= \frac{R}{2}\ln\left(\frac{1+\sin\beta}{1-\sin\beta}\right) = \frac{R}{2}\ln\left(\frac{1+\cos\phi\sin\omega}{1-\cos\phi\sin\omega}\right) \\
&= R\tanh^{-1}\left(\cos\phi\sin\omega\right)
\end{aligned} \tag{73}
$$

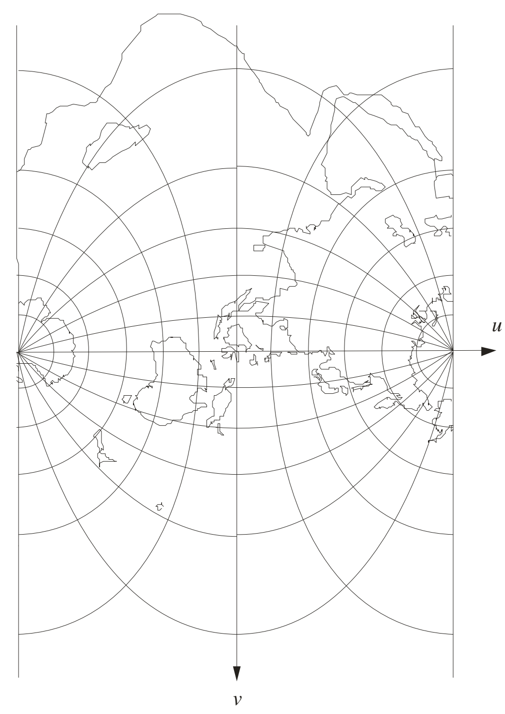

Figure 4 Transverse Mercator projection graticule interval 15°, central meridian $\lambda_0 = 120^\circ$ E

The transverse Mercator projection of the sphere is conformal, which can be verified by analysis of the Gaussian fundamental quantities *e, f, g* of the ϕ,λ spherical datum surface [see equations (64)] and *E, F, G* of the *u,v* projection surface.

For the projection surface $u = u(\phi,\omega)$, $v = v(\phi,\omega)$

$$\begin{aligned} E &= \left(\frac{\partial u}{\partial \phi}\right)^2 + \left(\frac{\partial v}{\partial \phi}\right)^2 \\ F &= \frac{\partial u}{\partial \phi}\frac{\partial u}{\partial \omega} + \frac{\partial v}{\partial \phi}\frac{\partial v}{\partial \omega} \\ G &= \left(\frac{\partial u}{\partial \omega}\right)^2 + \left(\frac{\partial v}{\partial \omega}\right)^2 \end{aligned} \tag{74}$$

Differentiating equations (73) noting that $\frac{d}{dx}\tan^{-1} y = \frac{1}{1+y^2}\frac{dy}{dx}$ and $\frac{d}{dx}\tanh^{-1} y = \frac{1}{1-y^2}\frac{dy}{dx}$ gives

$$\begin{aligned} \frac{\partial u}{\partial \phi} &= \frac{R\cos\omega}{1-\cos^2\phi\sin^2\omega}, \quad \frac{\partial u}{\partial \omega} = \frac{R\sin\phi\cos\phi\cos\omega}{1-\cos^2\phi\sin^2\omega} \\ \frac{\partial v}{\partial \phi} &= \frac{-R\sin\phi\sin\omega}{1-\cos^2\phi\sin^2\omega}, \quad \frac{\partial v}{\partial \omega} = \frac{R\cos\phi\cos\omega}{1-\cos^2\phi\sin^2\omega} \end{aligned} \tag{75}$$

and substituting these into equations (74) gives

$$E = \frac{R^2}{1-\cos^2\phi\sin^2\omega},\ F = 0,\ G = \frac{R^2\cos^2\phi}{1-\cos^2\phi\sin^2\omega} \tag{76}$$

Now using equations (30), (64) and (76) the scale factors *h* and *k* are equal and $f = F = 0$, then the projection is conformal.

Gauss-Lambert scale factor

Since the projection is conformal, the scale factor $m = h = k = \sqrt{E}/\sqrt{e}$ and

$$\begin{aligned} m &= \frac{1}{\sqrt{1-\cos^2\phi\sin^2\omega}} \\ &= 1 + \frac{1}{2}\cos^2\phi\sin^2\omega + \frac{3}{8}\cos^4\phi\sin^4\omega + \cdots \end{aligned} \tag{77}$$

Along the central meridian $\omega = 0$ and the central meridian scale factor $m_0 = 1$

Gauss-Lambert grid convergence

The grid convergence γ is the angle between the meridian and the grid-line parallel to the *u*-axis and is defined as

$$\tan\gamma = \left|\frac{dv}{du}\right| \tag{78}$$

and the total differentials *du* and *dv* are

$$du = \frac{\partial u}{\partial \phi}d\phi + \frac{\partial u}{\partial \omega}d\omega \ \text{ and } \ dv = \frac{\partial v}{\partial \phi}d\phi + \frac{\partial v}{\partial \omega}d\omega \tag{79}$$

Along a meridian ω is constant and $d\omega = 0$, and the grid convergence is obtained from

$$\tan\gamma = \left|\frac{\partial v}{\partial \phi}\Big/\frac{\partial u}{\partial \phi}\right| \tag{80}$$

and substituting partial derivatives from equations (75) gives

$$\gamma = \tan^{-1}(\sin\phi\tan\omega) \tag{81}$$

Transverse Mercator projection of the conformal sphere (Gauss-Schreiber projection)

The equations for the transverse Mercator projection of the conformal sphere are simply obtained by replacing spherical latitude ϕ with conformal latitude ϕ' in equations (73) and noting that the radius of the conformal sphere is *a* to give

$$\begin{aligned} u &= a\tan^{-1}\left(\frac{\tan\phi'}{\cos\omega}\right) \\ v &= a\tanh^{-1}(\cos\phi'\sin\omega) \end{aligned} \tag{82}$$

[These are Krüger's equations (36), §8, p. 20]

Alternatively, replacing ϕ with β and ω with θ in equations (67) then using equations (70), (72) and the identity (147); and finally replacing spherical latitude ϕ with conformal latitude ϕ' gives (Karney 2010)

$$\boxed{\begin{aligned} u &= a\tan^{-1}\left(\frac{\tan\phi'}{\cos\omega}\right) \\ v &= a\sinh^{-1}\left(\frac{\sin\omega}{\sqrt{\tan^2\phi' + \cos^2\omega}}\right) \end{aligned}} \tag{83}$$

$\tan\phi'$, which appears in both equations (83), can be evaluated by using hyperbolic and inverse hyperbolic functions (see Appendix) and equation (51) in the following manner.

Using equation (147) of the Appendix we may write

$$\ln\tan\left(\tfrac{1}{4}\pi + \tfrac{1}{2}\phi'\right) = \sinh^{-1}\tan\phi' \tag{84}$$

And the right-hand-side of equation (51) can be written as

$$\begin{aligned} &\ln\tan\left(\tfrac{1}{4}\pi + \tfrac{1}{2}\phi\right) - \tfrac{1}{2}\varepsilon\ln\left(\frac{1+\varepsilon\sin\phi}{1-\varepsilon\sin\phi}\right) \\ &= \sinh^{-1}\tan\phi - \varepsilon\tanh^{-1}(\varepsilon\sin\phi) \end{aligned} \tag{85}$$

Equating (84) and (85) gives

$$\sinh^{-1}\tan\phi' = \sinh^{-1}\tan\phi - \varepsilon\tanh^{-1}(\varepsilon\sin\phi) \tag{86}$$

With the substitution

$$\boxed{\sigma = \sinh\left\{\varepsilon\tanh^{-1}\left(\frac{\varepsilon\tan\phi}{\sqrt{1+\tan^2\phi}}\right)\right\}} \tag{87}$$

and some algebra, equation (86) can be rearranged as (Karney 2010)

$$\boxed{\tan\phi' = \tan\phi\sqrt{1+\sigma^2} - \sigma\sqrt{1+\tan^2\phi}} \tag{88}$$

The Gauss-Schreiber projection is also conformal since ϕ can be replaced by ϕ' in the previous analysis and $E/e = G/g$, and $f = F = 0$.

Gauss-Schreiber scale factor

The scale factor is given by equation (28) as

$$m^2 = \frac{(dS)^2}{(ds)^2} \tag{89}$$

For the datum surface (ellipsoid) equations (17) and (19) give (noting that $d\omega = d\lambda$)

$$(ds)^2 = \rho^2 (d\phi)^2 + \nu^2 \cos^2\phi (d\omega)^2 \tag{90}$$

For the projection plane

$$(dS)^2 = (du)^2 + (dv)^2 \tag{91}$$

$u = u(\phi,\omega)$ and $v = v(\phi,\omega)$ are given by equations (82); and the total differentials are

$$du = \frac{\partial u}{\partial\phi}d\phi + \frac{\partial u}{\partial\omega}d\omega \quad \text{and} \quad dv = \frac{\partial v}{\partial\phi}d\phi + \frac{\partial v}{\partial\omega}d\omega \tag{92}$$

Since the projection is conformal, scale is the same in all directions around any point. It is sufficient then to choose any one direction, say along a meridian where ω is constant and $d\omega = 0$. Hence

$$m^2 = \frac{1}{\rho^2}\left\{\left(\frac{\partial u}{\partial\phi}\right)^2 + \left(\frac{\partial v}{\partial\phi}\right)^2\right\} \tag{93}$$

The partial derivatives are evaluated using the chain rule for differentiation and equations (33), (54) and (82)

$$\frac{\partial u}{\partial\phi} = \frac{\partial u}{\partial\phi'}\frac{\partial\phi'}{\partial\psi}\frac{\partial\psi}{\partial\phi} \quad \text{and} \quad \frac{\partial v}{\partial\phi} = \frac{\partial v}{\partial\phi'}\frac{\partial\phi'}{\partial\psi}\frac{\partial\psi}{\partial\phi} \tag{94}$$

with

$$\begin{aligned}
\frac{\partial\psi}{\partial\phi} &= \frac{\varepsilon^2 - 1}{\left(1-\varepsilon^2\sin^2\phi\right)\cos\phi} \\
\frac{\partial\phi'}{\partial\psi} &= \frac{2\exp(\psi)}{1+\exp(2\psi)} = \cos\phi' \\
\frac{\partial u}{\partial\phi'} &= \frac{a\cos\omega}{1-\cos^2\phi'\sin^2\omega} \\
\frac{\partial v}{\partial\phi'} &= \frac{-a\sin\phi'\sin\omega}{1-\cos^2\phi'\sin^2\omega}
\end{aligned} \tag{95}$$

Substituting equations (95) into equations (94) and then into equation (93) and simplifying gives the scale factor m for the Gauss-Schreiber projection as

$$m = \frac{\sqrt{1+\tan^2\phi}\sqrt{1-\varepsilon^2\sin^2\phi}}{\sqrt{\tan^2\phi' + \cos^2\omega}} \tag{96}$$

Along the central meridian of the projection $\omega = 0$ and the central meridian scale factor m_0 is

$$\begin{aligned}
m_0 &= \frac{\cos\phi'}{\cos\phi}\sqrt{1-\varepsilon^2\sin^2\phi} \\
&= \frac{\cos\phi'}{\cos\phi}\left\{1 - \frac{1}{2}\varepsilon^2\sin^2\phi - \frac{1}{8}\varepsilon^4\sin^4\phi - \cdots\right\}
\end{aligned} \tag{97}$$

m_0 is not constant and varies slightly from unity, but a final conformal mapping from the Gauss-Schreiber u,v plane to an X,Y plane may be made and this final projection (the Gauss-Krüger projection) will have a constant scale factor along the central meridian.

Gauss-Schreiber grid convergence

The grid convergence for the Gauss-Schreiber projection is defined by equation (80) but the partial derivatives must be evaluated using the chain rule for differentiation [equations (94)] and

$$\tan\gamma = \left|\frac{\partial v}{\partial\phi}\Big/\frac{\partial u}{\partial\phi}\right| = \left|\frac{\partial v}{\partial\phi'}\Big/\frac{\partial u}{\partial\phi'}\right| \tag{98}$$

Using equations (95) the grid convergence for the Gauss-Schreiber projection is

$$\gamma = \tan^{-1}\left(\sin\phi'\tan\omega\right) = \tan^{-1}\left(\frac{\tan\phi'\tan\omega}{\sqrt{1+\tan^2\phi'}}\right) \tag{99}$$

Conformal mapping from the Gauss-Schreiber to the Gauss-Krüger projection

Using conformal mapping and complex functions (see Appendix), suppose that the mapping from the u,v plane of the Gauss-Schreiber projection (Figure 4) to the X,Y plane of the Gauss-Krüger projection (Figure 5) is given by

$$\frac{1}{A}(Y + iX) = f(u + iv) \tag{100}$$

where the Y-axis is the central meridian, the X-axis is the equator and A is the rectifying radius.

Let the complex function $f(u+iv)$ be

$$\begin{aligned}
f(u+iv) &= \frac{u}{a} + i\frac{v}{a} \\
&\quad + \sum_{r=1}^{\infty}\kappa_{2r}\sin\left(2r\left(\frac{u}{a}\right) + i\,2r\left(\frac{v}{a}\right)\right)
\end{aligned} \tag{101}$$

where a is the radius of the conformal sphere and κ_{2r} are as yet, unknown coefficients.

Expanding the complex trigonometric function in equation (101) gives

$$f\left(u+iv\right)=\frac{u}{a}+i\frac{v}{a}+\sum_{r=1}^{\infty}\kappa_{2r}\left(\begin{array}{l}\sin 2r\left(\frac{u}{a}\right)\cosh 2r\left(\frac{v}{a}\right)\\+i\cos 2r\left(\frac{u}{a}\right)\sinh 2r\left(\frac{v}{a}\right)\end{array}\right) \qquad (102)$$

and equating real and imaginary parts gives

$$\begin{aligned}\frac{Y}{A}&=\frac{u}{a}+\sum_{r=1}^{\infty}\kappa_{2r}\sin 2r\left(\frac{u}{a}\right)\cosh 2r\left(\frac{v}{a}\right)\\ \frac{X}{A}&=\frac{v}{a}+\sum_{r=1}^{\infty}\kappa_{2r}\cos 2r\left(\frac{u}{a}\right)\sinh 2r\left(\frac{v}{a}\right)\end{aligned} \qquad (103)$$

Now, along the central meridian $v=0$ and $\cosh 2v=\cosh 4v=\cdots=1$ and $\frac{Y}{A}$ in equation (103) becomes

$$\frac{Y}{A}=\frac{u}{a}+\kappa_2\sin 2\left(\frac{u}{a}\right)+\kappa_4\sin 4\left(\frac{u}{a}\right)+\kappa_6\sin 6\left(\frac{u}{a}\right)\cdots \qquad (104)$$

Furthermore, along the central meridian $\frac{u}{a}$ is an angular quantity that is identical to the conformal latitude ϕ' and equation (104) becomes

$$\frac{Y}{A}=\phi'+\kappa_2\sin 2\phi'+\kappa_4\sin 4\phi'+\kappa_6\sin 6\phi'\cdots \qquad (105)$$

Now, if the central meridian scale factor is unity then the *Y* coordinate is the meridian distance *M,* and $\frac{Y}{A}=\frac{M}{A}=\mu$ is the rectifying latitude and equation (105) becomes

$$\mu=\phi'+\kappa_2\sin 2\phi'+\kappa_4\sin 4\phi'+\kappa_6\sin 6\phi'\cdots \qquad (106)$$

This equation is identical in form to equation (62) and we may conclude that the coefficients $\{\kappa_{2r}\}$ are equal to the coefficients $\{\alpha_{2r}\}$ in equation (62); and the Gauss-Krüger projection is given by

$$\boxed{\begin{aligned}X&=A\left\{\frac{v}{a}+\sum_{r=1}^{\infty}\alpha_{2r}\cos 2r\left(\frac{u}{a}\right)\sinh 2r\left(\frac{v}{a}\right)\right\}\\ Y&=A\left\{\frac{u}{a}+\sum_{r=1}^{\infty}\alpha_{2r}\sin 2r\left(\frac{u}{a}\right)\cosh 2r\left(\frac{v}{a}\right)\right\}\end{aligned}} \qquad (107)$$

[These are Krüger's equations (42), §8, p. 21.]

A is given by equation (40), $\frac{u}{a}$ and $\frac{v}{a}$ are given by equations (83) and we have elected to use coefficients α_{2r} up to $r=8$ given by equations (63).

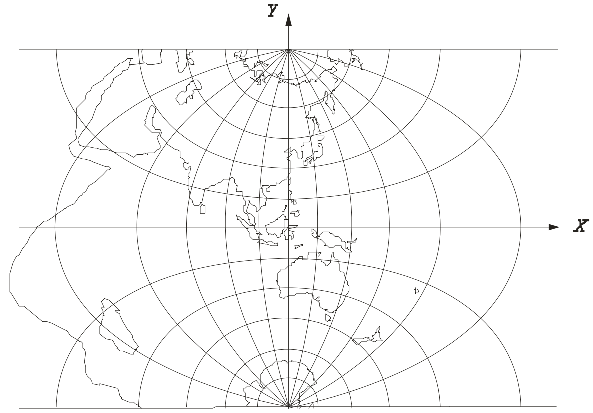

Figure 5 Gauss-Krüger projection
graticule interval 15°, central meridian $\lambda_0=120^\circ$ E

[Note that the graticules of Figures 4 and 5 are for different projections but are indistinguishable at the printed scales and for the longitude extent shown. If a larger eccentricity was chosen, say $\varepsilon=\frac{1}{10}$ $(f\approx 1/199.5)$ and the mappings scaled so that the distances from the equator to the pole were identical, there would be some noticeable differences between the graticules at large distances from the central meridian. One of the authors (Karney 2010, Fig. 1) has examples of these graticule differences.]

Finally, *X* and *Y* are scaled and shifted to give *E* (east) and *N* (north) coordinates related to a false origin

$$\boxed{\begin{aligned}E&=m_0X+E_0\\ N&=m_0Y+N_0\end{aligned}} \qquad (108)$$

m_0 is the central meridian scale factor and the quantities E_0, N_0 are offsets that make the *E,N* coordinates positive in the area of interest. The origin of the *X,Y* coordinates is at the intersection of the equator and the central meridian and is known as the true origin. The origin of the *E,N* coordinates is known as the false origin and it is located at $X=-E_0$, $Y=-N_0$.

Gauss-Krüger scale factor

The scale factor for the Gauss-Krüger projection can be derived in a similar way to the derivation of the scale factor for the Gauss-Schreiber projection and we have

$$\begin{aligned}(dS)^2&=(dX)^2+(dY)^2\\ (ds)^2&=\rho^2(d\phi)^2+\upsilon^2\cos^2\phi\,(d\omega)^2\end{aligned}$$

where $X=X(u,v)$, $Y=Y(u,v)$ and the total differentials *dX* and *dY* are

$$\begin{aligned}dX&=\frac{\partial X}{\partial u}du+\frac{\partial X}{\partial v}dv\\ dY&=\frac{\partial Y}{\partial u}du+\frac{\partial Y}{\partial v}dv\end{aligned} \qquad (109)$$

du and *dv* are given by equations (92) and substituting these into equations (109) gives

$$dX = \frac{\partial X}{\partial u}\left\{\frac{\partial u}{\partial \phi}d\phi + \frac{\partial u}{\partial \omega}d\omega\right\} + \frac{\partial X}{\partial v}\left\{\frac{\partial v}{\partial \phi}d\phi + \frac{\partial v}{\partial \omega}d\omega\right\}$$
$$dY = \frac{\partial Y}{\partial u}\left\{\frac{\partial u}{\partial \phi}d\phi + \frac{\partial u}{\partial \omega}d\omega\right\} + \frac{\partial Y}{\partial v}\left\{\frac{\partial v}{\partial \phi}d\phi + \frac{\partial v}{\partial \omega}d\omega\right\}$$

Choosing to evaluate the scale along a meridian where ω is constant and $d\omega = 0$ gives

$$\begin{aligned} dX &= \left(\frac{\partial X}{\partial u}\frac{\partial u}{\partial \phi} + \frac{\partial X}{\partial v}\frac{\partial v}{\partial \phi}\right)d\phi \\ dY &= \left(\frac{\partial Y}{\partial u}\frac{\partial u}{\partial \phi} + \frac{\partial Y}{\partial v}\frac{\partial v}{\partial \phi}\right)d\phi \end{aligned} \tag{110}$$

and

$$m^2 = \frac{(dS)^2}{(ds)^2} = \frac{(dX)^2 + (dY)^2}{\rho^2 (d\phi)^2} \tag{111}$$

Differentiating equations (107) gives

$$\frac{\partial X}{\partial u} = \frac{A}{a}q,\ \frac{\partial X}{\partial v} = \frac{A}{a}p,\ \frac{\partial Y}{\partial u} = \frac{\partial X}{\partial v},\ \frac{\partial Y}{\partial v} = -\frac{\partial X}{\partial u} \tag{112}$$

where

$$\boxed{\begin{aligned} q &= -\sum_{r=1}^{\infty} 2r\alpha_{2r} \sin 2r\left(\frac{u}{a}\right)\sinh 2r\left(\frac{v}{a}\right) \\ p &= 1 + \sum_{r=1}^{\infty} 2r\alpha_{2r} \cos 2r\left(\frac{u}{a}\right)\cosh 2r\left(\frac{v}{a}\right) \end{aligned}} \tag{113}$$

Substituting equations (112) into (110) and then into the equation (111) and simplifying gives

$$m^2 = \left(\frac{A}{a}\right)^2 \left(q^2 + p^2\right)\left\{\frac{1}{\rho^2}\left(\left(\frac{\partial u}{\partial \phi}\right)^2 + \left(\frac{\partial v}{\partial \phi}\right)^2\right)\right\} \tag{114}$$

The term in braces $\{\ \}$ is the square of the scale factor for the Gauss-Schreiber projection [see equation (93)] and so, using equation (96), we may write the scale factor for the Gauss-Krüger projection as

$$\boxed{m = m_0\left(\frac{A}{a}\right)\sqrt{q^2 + p^2}\left\{\frac{\sqrt{1+\tan^2\phi}\sqrt{1-\varepsilon^2\sin^2\phi}}{\sqrt{\tan^2\phi' + \cos^2\omega}}\right\}} \tag{115}$$

where m_0 is the central meridian scale factor, q and p are found from equations (113), $\tan\phi'$ from equation (88) and A from equation (40).

Gauss-Krüger grid convergence

The grid convergence for the Gauss-Krüger projection is defined by

$$\tan\gamma = \left|\frac{dX}{dY}\right| \tag{116}$$

Using equations (110) and (112) we may write equation (116) as

$$\tan\gamma = \left|\frac{q\dfrac{\partial u}{\partial \phi} + p\dfrac{\partial v}{\partial \phi}}{p\dfrac{\partial u}{\partial \phi} - q\dfrac{\partial v}{\partial \phi}}\right| = \left|\frac{\dfrac{q}{p} + \left(\dfrac{\partial v}{\partial \phi}\Big/\dfrac{\partial u}{\partial \phi}\right)}{1 - \dfrac{q}{p}\left(\dfrac{\partial v}{\partial \phi}\Big/\dfrac{\partial u}{\partial \phi}\right)}\right| \tag{117}$$

Let $\gamma = \gamma_1 + \gamma_2$, then using a trigonometric addition formula write

$$\tan\gamma = \tan(\gamma_1 + \gamma_2) = \frac{\tan\gamma_1 + \tan\gamma_2}{1 - \tan\gamma_1\tan\gamma_2} \tag{118}$$

Noting the similarity between equations (117) and (118) we may define

$$\tan\gamma_1 = \left|\frac{q}{p}\right| \quad \text{and} \quad \tan\gamma_2 = \left|\frac{\partial v}{\partial \phi}\Big/\frac{\partial u}{\partial \phi}\right| \tag{119}$$

and γ_2 is the grid convergence on the Gauss-Schreiber projection [see equations (98) and (99)]. So the grid convergence on the Gauss-Krüger projection is

$$\boxed{\gamma = \tan^{-1}\left(\frac{q}{p}\right) + \tan^{-1}\left(\frac{\tan\phi'\tan\omega}{\sqrt{1+\tan^2\phi'}}\right)} \tag{120}$$

Conformal mapping from the Gauss-Krüger projection to the ellipsoid

The conformal mapping from the Gauss-Krüger projection to the ellipsoid is achieved in three steps:

(i) A conformal mapping from the Gauss-Krüger to the Gauss-Schreiber projection giving *u,v* coordinates, then

(ii) Solving for $\tan\phi'$ and $\tan\omega$ given the *u,v* Gauss-Schreiber coordinates from which $\lambda = \lambda_0 \pm \omega$, and finally

(iii) Solving for $\tan\phi$ by Newton-Raphson iteration and then obtaining ϕ.

The development of the equations for these three steps is set out below.

Gauss-Schreiber coordinates from Gauss-Krüger coordinates

In a similar manner as outlined above, suppose that the mapping from the *X,Y* plane of the Gauss-Krüger projection to the *u,v* plane of the Gauss-Schreiber projection is given by the complex function

$$\frac{1}{a}(u + iv) = F(Y + iX) \tag{121}$$

If *E,N* coordinates are given and E_0, N_0 and m_0 are known, then from equations (108)

$$\boxed{X = \frac{E - E_0}{m_0} \quad \text{and} \quad Y = \frac{N - N_0}{m_0}} \tag{122}$$

Let the complex function $F(Y+iX)$ be

$$F(Y+iX)=\frac{Y}{A}+i\frac{X}{A}+\sum_{r=1}^{\infty}K_{2r}\sin\left(2r\left(\frac{Y}{A}\right)+i\,2r\left(\frac{X}{A}\right)\right) \tag{123}$$

where A is the rectifying radius and K_{2r} are as yet unknown coefficients.

Expanding the complex trigonometric function in equation (123) and then equating real and imaginary parts gives

$$\begin{aligned}\frac{u}{a}&=\frac{Y}{A}+\sum_{r=1}^{\infty}K_{2r}\sin 2r\left(\frac{Y}{A}\right)\cosh 2r\left(\frac{X}{A}\right)\\ \frac{v}{a}&=\frac{X}{A}+\sum_{r=1}^{\infty}K_{2r}\cos 2r\left(\frac{Y}{A}\right)\sinh 2r\left(\frac{X}{A}\right)\end{aligned} \tag{124}$$

Along the central meridian $\frac{Y}{A}=\frac{M}{A}=\mu$ the rectifying latitude and $X=0$ and $\cosh(0)=1$. Also, $\frac{u}{a}$ is an angular quantity that is identical to the conformal latitude ϕ' and we may write the first of equations (124) as

$$\phi'=\mu+K_2\sin 2\mu+K_4\sin 4\mu+K_6\sin 6\mu+\cdots \tag{125}$$

This equation is identical in form to equation (60) and we may conclude that the coefficients $\{K_{2r}\}$ are equal to the coefficients $\{\beta_{2r}\}$ in equation (60) and the ratios $\frac{u}{a}$ and $\frac{v}{a}$ are given by

$$\boxed{\begin{aligned}\frac{u}{a}&=\frac{Y}{A}+\sum_{r=1}^{\infty}\beta_{2r}\sin 2r\left(\frac{Y}{A}\right)\cosh 2r\left(\frac{X}{A}\right)\\ \frac{v}{a}&=\frac{X}{A}+\sum_{r=1}^{\infty}\beta_{2r}\cos 2r\left(\frac{Y}{A}\right)\sinh 2r\left(\frac{X}{A}\right)\end{aligned}} \tag{126}$$

where A is given by equation (40) and we have elected to use coefficients β_{2r} up to $r=8$ given by equations (61).

Conformal latitude and longitude difference from Gauss-Schreiber coordinates

Equations (83) can be re-arranged and solved for $\tan\phi'$ and $\tan\omega$ as functions of the ratios $\frac{u}{a}$ and $\frac{v}{a}$ giving

$$\boxed{\begin{aligned}\tan\phi'&=\frac{\sin\left(\frac{u}{a}\right)}{\sqrt{\sinh^2\left(\frac{v}{a}\right)+\cos^2\left(\frac{u}{a}\right)}}\\ \tan\omega&=\sinh\left(\frac{v}{a}\right)\Big/\cos\left(\frac{u}{a}\right)\end{aligned}} \tag{127}$$

Solution for latitude by Newton-Raphson iteration

To evaluate $\tan\phi$ after obtaining $\tan\phi'$ from equation (127) consider equations (87) and (88) with the substitutions $t=\tan\phi$ and $t'=\tan\phi'$

$$t'=t\sqrt{1+\sigma^2}-\sigma\sqrt{1+t^2} \tag{128}$$

and
$$\sigma=\sinh\left\{\varepsilon\tanh^{-1}\left(\frac{\varepsilon t}{\sqrt{1+t^2}}\right)\right\} \tag{129}$$

$t=\tan\phi$ can be evaluated using the Newton-Raphson method for the real roots of the equation $f(t)=0$ given in the form of an iterative equation

$$\boxed{t_{n+1}=t_n-\frac{f(t_n)}{f'(t_n)}} \tag{130}$$

where t_n denotes the n^{th} iterate and $f(t)$ is given by

$$f(t)=t\sqrt{1+\sigma^2}-\sigma\sqrt{1+t^2}-t' \tag{131}$$

The derivative $f'(t)=\frac{d}{dt}\{f(t)\}$ is given by

$$f'(t)=\left(\sqrt{1+\sigma^2}\sqrt{1+t^2}-\sigma t\right)\frac{(1-\varepsilon^2)\sqrt{1+t^2}}{1+(1-\varepsilon^2)t^2} \tag{132}$$

where $t'=\tan\phi'$ is fixed.

An initial value for t_1 can be taken as $t_1=t'=\tan\phi'$ and the functions $f(t_1)$ and $f'(t_1)$ evaluated from equations (129), (131) and (132). t_2 is now computed from equation (130) and this process repeated to obtain $t_3,t_4,\ldots$. This iterative process can be concluded when the difference between t_{n+1} and t_n reaches an acceptably small value, and then the latitude is given by $\phi=\tan^{-1}t_{n+1}$.

This concludes the development of the Gauss-Krüger projection.

TRANSFORMATIONS BETWEEN THE ELLIPSOID AND THE GAUSS-KRÜGER PLANE

Forward transformation: $\phi,\lambda\rightarrow X,Y$ given a,f,λ_0,m_0

1. Compute ellipsoid constants ε^2, n and powers $n^2,n^3,\ldots,n^8$
2. Compute the rectifying radius A from equation (40)
3. Compute conformal latitude ϕ' from equations (87) and (88)
4. Compute longitude difference $\omega=\lambda-\lambda_0$
5. Compute the u,v Gauss-Schreiber coordinates from equations (83)
6. Compute the coefficients $\{\alpha_{2r}\}$ from equations (63)
7. Compute X,Y coordinates from equations (107)
8. Compute q and p from equations (113)

9. Compute scale factor m from equation (115)
10. Compute grid convergence γ from equation (120)

Inverse transformation: $X, Y \to \phi, \lambda$ given a, f, λ_0, m_0

1. Compute ellipsoid constants ε^2, n and powers $n^2, n^3, \ldots, n^8$
2. Compute the rectifying radius A from equation (40)
3. Compute the coefficients $\{\beta_{2r}\}$ from equations (61)
4. Compute the ratios $\frac{u}{a}, \frac{v}{a}$ from equations (126)
5. Compute conformal latitude ϕ' and longitude difference ω from equations (127)
6. Compute $t = \tan\phi$ by Newton-Raphson iteration using equations (130), (131) and (132)
7. Compute latitude $\phi = \tan^{-1} t$ and longitude $\lambda = \lambda_0 \pm \omega$
8. Compute the coefficients $\{\alpha_n\}$ from equations (63)
9. Compute q and p from equations (113)
10. Compute scale factor m from equation (115)
11. Compute grid convergence γ from equation (120)

ACCURACY OF THE TRANSFORMATIONS

One of the authors (Karney, 2010) has compared Krüger's series to order n^8 (set out above) with an exact transverse Mercator projection defined by Lee (1976) and shows that errors in positions computed from this series are less than 5 nanometres anywhere within a distance of 4200 km of the central meridian (equivalent to $\omega \approx 37.7^\circ$ at the equator). So we can conclude that Kruger's series (to order n^8) is easily capable of micrometre precision within 30° of a central meridian.

THE 'OTHER' GAUSS-KRÜGER PROJECTION

In Krüger's original work (Krüger 1912) of 172 pages (plus vii pages), Krüger develops the mapping equations shown above in 22 pages, with a further 14 pages of examples of the forward and inverse transformations. In the next 38 pages Krüger develops and explains an alternative approach: direct transformations from the ellipsoid to the plane and from the plane to the ellipsoid. The remaining 100 pages are concerned with the intricacies of the geodesic projected on the transverse Mercator plane, arc-to-chord, line scale factor, etc.

This alternative approach is outlined in the Appendix and for the forward transformation [see equations (160)] the equations involve functions containing powers of the longitude difference $\omega^2, \omega^3, \ldots$ and derivatives $\frac{dM}{d\psi}$, $\frac{d^2M}{d\psi^2}$, $\frac{d^3M}{d\psi^3}, \ldots$ For the inverse transformation [see equations (165) and (167)] the equations involve powers of the X coordinate $X^2, X^3, X^4, \ldots$ and derivatives $\frac{d\phi_1}{d\psi_1}$, $\frac{d^2\phi_1}{d\psi_1^2}$, $\frac{d^3\phi_1}{d\psi_1^3}, \ldots$ and $\frac{d\psi_1}{dY}$, $\frac{d^2\psi_1}{dY^2}$, $\frac{d^3\psi_1}{dY^3}, \ldots$ For both transformations, the higher order derivatives become excessively complicated and are not generally known (or approximated) beyond the eighth derivative.

Redfearn (1948) and Thomas (1952) both derive identical formulae, extending (slightly) Kruger's equations, and updating the notation and formulation. These formulae are regarded as the standard for transformations between the ellipsoid and the transverse Mercator projection. For example, GeoTrans (2010) uses Thomas' equations and Geoscience Australia defines Redfearn's equations as the method of transformation between the Geocentric Datum of Australia (ellipsoid) and Map Grid Australia (transverse Mercator) [GDAV2.3].

The apparent attractions of these formulae are:

(i) their wide-spread use and adoption by government mapping authorities, and
(ii) there are no hyperbolic functions.

The weakness of these formulae are:

(a) they are only accurate within relatively small bands of longitude difference about the central meridian (mm accuracy for $\omega < 6^\circ$) and
(b) at large longitude differences $(\omega > 30^\circ)$ they can give wildly inaccurate results (1-2 km errors).

The inaccuracies in Redfearn's (and Thomas's) equations are most evident in the inverse transformation $X, Y \to \phi, \omega$. Table 1 shows a series of points each having latitude $\phi = 75^\circ$ but with increasing longitude differences ω from a central meridian. The X, Y coordinates are computed using Krüger's series and can be regarded as exact (at mm accuracy) and the column headed Readfearn ϕ, ω are the values obtained from Redfearn's equations for the inverse transformation. The error is the distance on the ellipsoid between the given ϕ, ω in the first column and the Redfearn ϕ, ω in the third column.

The values in the table have been computed for the GRS80 ellipsoid ($a = 6378137$ m , $f = 1/298.257222101$) with $m_0 = 1$

point	Gauss-Krüger	Redfearn	error
φ 75° ω 6°	X 173137.521 Y 8335703.234	φ 75° 00' 00.0000" ω 5° 59' 59.9999"	0.001
φ 75° ω 10°	X 287748.837 Y 8351262.809	φ 75° 00' 00.0000" ω 9° 59' 59.9966"	0.027
φ 75° ω 15°	X 429237.683 Y 8381563.943	φ 75° 00' 00.0023" ω 14° 59' 59.8608"	1.120
φ 75° ω 20°	X 567859.299 Y 8423785.611	φ 75° 00' 00.0472" ω 19° 59' 57.9044"	16.888
φ 75° ω 30°	X 832650.961 Y 8543094.338	φ 75° 00' 03.8591" ω 29° 58' 03.5194"	942.737
φ 75° ω 35°	X 956892.903 Y 8619555.491	φ 75° 00' 23.0237" ω 34° 49' 57.6840"	4.9 km

Table 1

This problem is highlighted when considering a map of Greenland (Figure 6), which is almost the ideal 'shape' for a transverse Mercator projection, having a small east-west extent (approx. 1650 km) and large north-south extent (approx. 2600 km).

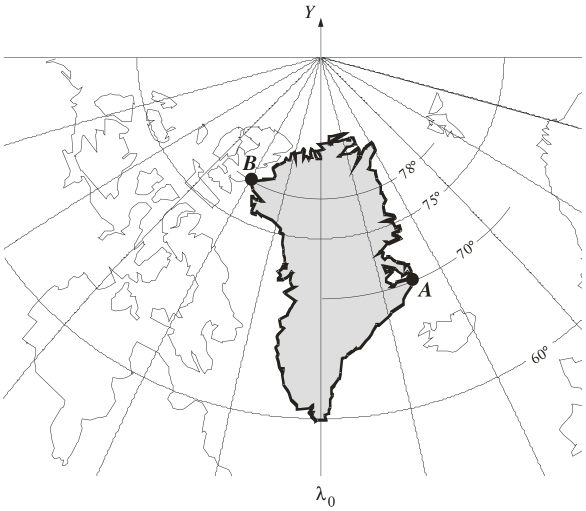

Figure 6 Gauss-Krüger projection of Greenland graticule interval 15°, central meridian $\lambda_0 = 45^\circ$ W

Points *A* and *B* represent two extremes if a central meridian is chosen as $\lambda_0 = 45^\circ$ W . *A* ($\phi = 70^\circ$ N , $\lambda = 22^\circ 30'$W) is a point furthest from the central meridian (approx. 850 km); and *B* ($\phi = 78^\circ$ N $\lambda = 75^\circ$ W) would have the greatest west longitude.

Table 2 shows the errors at these points for the GRS80 ellipsoid with $m_0 = 1$ for the inverse transformation using Redfearn's equations.

point	Gauss-Krüger	Redfearn	error
A φ 70° ω 22.5°	X 842115.901 Y 7926858.314	φ 75° 00' 00.2049" ω 22° 29' 53.9695"	64.282
B φ 78° ω -30°	X -667590.239 Y 8837145.459	φ 78° 00' 03.1880" ω-29° 57' 59.2860"	784.799

Table 2

CONCLUSION

We have provided here a reasonably complete derivation of the Gauss-Krüger projection equations that allow micrometre accuracy in the forward and inverse mappings between the ellipsoid and plane. And we have provided some commentary on the 'other' Gauss-Krüger equations in wide use in the geospatial community. These other equations offer only limited accuracy and should be abandoned in favour of the equations (and methods) we have outlined.

Our work is not original; indeed these equations were developed by Krüger almost a century ago. But with the aid of computer algebra systems we have extended Krüger's series – as others have done, e.g. Engsager & Poder (2007) – so that the method is capable of very high accuracy at large distances from a central meridian. This makes the transverse Mercator projection a much more useful projection for the geospatial community.

We also hope that this paper may be useful to mapping organisations wishing to 'upgrade' transformation software that use formulae given by Redfearn (1948) or Thomas (1952) – they are unnecessarily inaccurate.

NOMENCLATURE

This paper		Krüger
α_k	coefficients in series for μ	γ
β_k	coefficients in series for ϕ'	β
β	oblique latitude	
γ	grid convergence	c
ε	eccentricity of ellipsoid	e
ε^2	eccentricity of ellipsoid squared	e^2
ε'^2	2nd eccentricity of ellipsoid squared	
θ	oblique longitude	
λ	longitude	L
λ_0	longitude of central meridian	
μ	rectifying latitude	Ψ
ρ	radius of curvature (meridian)	R
υ	radius of curvature (prime vertical)	N
σ	function of latitude	
ϕ	latitude	B
ϕ'	conformal latitude	b
ψ	isometric latitude	ω
ω	longitude difference: $\omega = \lambda - \lambda_0$	
	angle between parametric curves	
A	rectifying radius	A
a	semi-major axis of ellipsoid	a
b	semi-minor axis of ellipsoid	b
c	polar radius of curvature	
ds	differential distance on datum surface	
dS	differential distance on projection surface	
e	Gaussian fundamental quantity	
f	flattening of ellipsoid	
	Gaussian fundamental quantity	
g	Gaussian fundamental quantity	
h	scale factor along meridian	
k	scale factor along parallel	
M	meridian distance	X
m	scale factor	m
m_0	central meridian scale factor	m_0
n	3rd flattening of ellipsoid	n
Q	quadrant length of ellipsoid	$\mathfrak{M}$
s	distance on ellipsoid	s
t	$t = \tan\phi$	
t'	$t' = \tan\phi'$	
u	transverse Mercator coordinate	ξ'
V	ellipsoid latitude function	Q
v	transverse Mercator coordinate	η'
W	ellipsoid latitude function	

APPENDIX

Reversion of a series

If we have an expression for a variable *z* as a series of powers or functions of another variable *y* then we may, by a reversion of the series, find an expression for *y* as series of functions of *z*. Reversion of a series can be done using Lagrange's theorem, a proof of which can be found in Bromwich (1991).

Suppose that

$$y = z + xF(y) \quad \text{or} \quad z = y - xF(y) \tag{133}$$

then Lagrange's theorem states that for any *f*

$$\begin{aligned} f(y) = f(z) &+ \frac{x}{1!}F(z)f'(z) \\ &+ \frac{x^2}{2!}\frac{d}{dz}\left[\{F(z)\}^2 f'(z)\right] \\ &+ \frac{x^3}{3!}\frac{d^2}{dz^2}\left[\{F(z)\}^3 f'(z)\right] \\ &+ \cdots \\ &+ \frac{x^n}{n!}\frac{d^{n-1}}{dz^{n-1}}\left[\{F(z)\}^n f'(z)\right] \\ &+ \cdots \end{aligned} \tag{134}$$

As an example, consider the series for rectifying latitude μ

$$\mu = \phi + d_2 \sin 2\phi + d_4 \sin 4\phi + d_6 \sin 6\phi + \cdots \tag{135}$$

And we wish to find an expression for ϕ as a function of μ.

Comparing the variables in equations (135) and (133), $z = \mu$, $y = \phi$ and $x = -1$; and if we choose $f(y) = y$ then $f(z) = z$ and $f'(z) = 1$. So equation (135) can be expressed as

$$\mu = \phi + F(\phi) \tag{136}$$

and Lagrange's theorem gives

$$\begin{aligned} \phi = \mu - F(\mu) &+ \frac{1}{2}\frac{d}{d\mu}\left[\{F(\mu)\}^2\right] \\ &- \frac{1}{6}\frac{d^2}{d\mu^2}\left[\{F(\mu)\}^3\right] + \cdots \\ &+ \frac{(-1)^n}{n!}\frac{d^{n-1}}{d\mu^{n-1}}\left[\{F(\mu)\}^n\right] \end{aligned} \tag{137}$$

where

$$F(\phi) = d_2 \sin 2\phi + d_4 \sin 4\phi + d_6 \sin 6\phi + \cdots$$

and so

$$F(\mu) = d_2 \sin 2\mu + d_4 \sin 4\mu + d_6 \sin 6\mu + \cdots$$

Taylor's theorem

This theorem, due to the English mathematician Brook Taylor (1685–1731) enables a function $f(x)$ near a point $x = a$ to be expressed from the values $f(a)$ and the successive derivatives of $f(x)$ evaluated at $x = a$.

Taylor's polynomial may be expressed in the following form

$$\begin{aligned} f(x) = f(a) + (x-a)f'(a) &+ \frac{(x-a)^2}{2!}f''(a) \\ &+ \frac{(x-a)^3}{3!}f'''(a) + \cdots \\ &+ \frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + R_n \end{aligned} \tag{138}$$

where R_n is the remainder after *n* terms and $f'(a)$, $f''(a), \ldots$ etc. are derivatives of the function $f(x)$ evaluated at $x = a$.

Taylor's theorem can also be expressed as power series

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k \tag{139}$$

where $f^{(k)}(a) \equiv \left[\frac{d^k}{dx^k} f(x)\right]_{x=a}$

As an example of the use of Taylor's theorem, suppose we have an expression for the difference between latitude ϕ and the rectifying latitude μ [see equation (45)]

$$\phi - \mu = D_2 \sin 2\mu + D_4 \sin 4\mu + D_6 \sin 6\mu + \cdots \tag{140}$$

and we wish to find expressions for $\sin 2\phi$, $\sin 4\phi$, $\sin 6\phi$, etc. as functions of μ.

We can use Taylor's theorem to find an expression for $f(\phi) = \sin\phi$ about $\phi = \mu$ as

$$\begin{aligned} \sin\phi = \sin\mu &+ (\phi-\mu)\frac{d}{d\phi}\sin\phi\Big|_{\phi=\mu} \\ &+ \frac{1}{2!}(\phi-\mu)^2 \frac{d^2}{d\phi^2}\sin\phi\Big|_{\phi=\mu} \\ &+ \frac{1}{3!}(\phi-\mu)^3 \frac{d^3}{d\phi^3}\sin\phi\Big|_{\phi=\mu} \\ &+ \cdots \end{aligned}$$

giving

$$\begin{aligned} \sin\phi = \sin\mu + (\phi-\mu)\cos\mu &- \frac{(\phi-\mu)^2}{2}\sin\mu \\ - \frac{(\phi-\mu)^3}{6}\cos\mu &+ \frac{(\phi-\mu)^4}{24}\sin\mu + \cdots \end{aligned} \tag{141}$$

Replacing ϕ with 2ϕ and μ with 2μ in equation (141) and substituting $\phi - \mu$ from equation (140) gives an expression for $\sin 2\phi$. Using similar replacements and substitutions, expressions for $\sin 4\phi$, $\sin 6\phi$, etc. can be developed.

Hyperbolic functions

The basic functions are the hyperbolic sine of x, denoted by $\sinh x$, and the hyperbolic cosine of x denoted by $\cosh x$; they are defined as

$$\begin{aligned}\sinh x &= \frac{\exp(x)-\exp(-x)}{2}\\ \cosh x &= \frac{\exp(x)+\exp(-x)}{2}\end{aligned} \tag{142}$$

Other hyperbolic functions are in terms of these

$$\begin{aligned}\tanh x &= \frac{\sinh x}{\cosh x}, \quad \coth x = \frac{1}{\tanh x}\\ \operatorname{sech} x &= \frac{1}{\cosh x}, \quad \operatorname{cosech} x = \frac{1}{\sinh x}\end{aligned} \tag{143}$$

The inverse hyperbolic function of $\sinh x$ is $\sinh^{-1} x$ and is defined by $\sinh^{-1}(\sinh x) = x$. Similarly $\cosh^{-1} x$ and $\tanh^{-1} x$ are defined by $\cosh^{-1}(\cosh x) = x$ and $\tanh^{-1}(\tanh x) = x$; both requiring $x > 0$ and as a consequence of the definitions

$$\begin{aligned}\sinh^{-1} x &= \ln\left(x+\sqrt{x^2+1}\right) & -\infty < x < \infty\\ \cosh^{-1} x &= \ln\left(x+\sqrt{x^2-1}\right) & x \ge 1\\ \tanh^{-1} x &= \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) & -1 < x < 1\end{aligned} \tag{144}$$

A useful identity linking circular and hyperbolic functions used in conformal mapping is obtained by considering the following.

Using the trigonometric addition and double angle formula we have

$$\begin{aligned}\ln\tan\left(\tfrac{1}{4}\pi+\tfrac{1}{2}x\right) &= \ln\frac{\cos\frac{1}{2}x+\sin\frac{1}{2}x}{\cos\frac{1}{2}x-\sin\frac{1}{2}x}\\ &= \ln\frac{\left(\cos\frac{1}{2}x+\sin\frac{1}{2}x\right)^2}{\cos^2\frac{1}{2}x-\sin^2\frac{1}{2}x} = \ln\frac{1+\sin x}{\cos x}\end{aligned} \tag{145}$$

Also, replacing x with $\tan x$ in the definition of the inverse hyperbolic functions in equations (144) we have

$$\begin{aligned}\sinh^{-1}\tan x &= \ln\left(\tan x+\sqrt{1+\tan^2 x}\right)\\ &= \ln\left(\tan x+\sec x\right) = \ln\frac{1+\sin x}{\cos x}\end{aligned} \tag{146}$$

And equating $\ln\frac{1+\sin x}{\cos x}$ from equations (145) and (146) gives

$$\ln\tan\left(\tfrac{1}{4}\pi+\tfrac{1}{2}x\right) = \sinh^{-1}\tan x \tag{147}$$

Conformal mapping and complex functions

A theory due to Gauss states that a conformal mapping from the ψ,ω datum surface to the X,Y projection surface can be represented by the complex expression

$$Y + iX = f\left(\psi + i\omega\right) \tag{148}$$

Providing that ψ and ω are isometric parameters and the complex function $f\left(\psi+i\omega\right)$ is analytic. $i=\sqrt{-1}$ (the imaginary number), and the left-hand side of equation (148) is a complex number consisting of a real and imaginary part. The right-hand-side of equation (148) is a complex function, i.e., a function of real and imaginary parameters ψ and ω respectively. The complex function $f\left(\psi+i\omega\right)$ is analytic if it is everywhere differentiable and we may think of an analytic function as one that describes a smooth surface having no holes, edges or discontinuities.

Part of a necessary and sufficient condition for $f\left(\psi+i\omega\right)$ to be analytic is that the Cauchy-Riemann equations are satisfied, i.e., (Sokolnikoff & Redheffer 1966)

$$\frac{\partial Y}{\partial \psi} = \frac{\partial X}{\partial \omega} \quad \text{and} \quad \frac{\partial Y}{\partial \omega} = -\frac{\partial X}{\partial \psi} \tag{149}$$

As an example, consider the Mercator projection of the sphere shown in Figure 2 where the conformal mapping from the sphere (datum surface) to the plane is given by equations (67) and using the isometric latitude given by equation (34) the mapping equations are

$$\begin{aligned}X &= R\left(\lambda-\lambda_0\right) = R\,\omega\\ Y &= R\ln\tan\left(\tfrac{1}{4}\pi+\tfrac{1}{2}\phi\right) = R\psi\end{aligned} \tag{150}$$

These equations can be expressed as the complex equation

$$Z = Y + iX = R\left(\psi + i\omega\right) \tag{151}$$

where Z is a complex function defining the Mercator projection.

The transverse Mercator projection of the sphere shown in Figure 4 can also be expressed as a complex equation. Using the identity (147) and equation (150) we may define the transverse Mercator projection by

$$Z = Y + iX = R\left(\sinh^{-1}\tan\phi + i\omega\right) \tag{152}$$

Now suppose we have another complex function

$$w = u + iv = \tan^{-1}\sinh Z \tag{153}$$

representing a conformal transformation from the X,Y plane to the u,v plane.

What are the functions *u* and *v*?

It turns out, after some algebra, *u* and *v* are of the same form as equations (83)

$$\begin{aligned} u &= R\tan^{-1}\left(\frac{\tan\phi}{\cos\omega}\right) \\ v &= R\sinh^{-1}\left(\frac{\sin\omega}{\sqrt{\tan^2\phi+\cos^2\omega}}\right) \end{aligned} \tag{154}$$

and the transverse Mercator projection of the sphere is defined by the complex function

$$\begin{aligned} w &= u+iv \\ &= R\left\{\begin{aligned} &\tan^{-1}\left(\frac{\tan\phi}{\cos\omega}\right) \\ &+i\sinh^{-1}\left(\frac{\sin\omega}{\sqrt{\tan^2\phi+\cos^2\omega}}\right) \end{aligned}\right\} \end{aligned} \tag{155}$$

Other complex functions achieve the same result. For example Lauf (1983) shows that

$$w = u+iv = 2R\left\{\tan^{-1}\exp\left(\psi+i\omega\right)-\tfrac{1}{4}\pi\right\} \tag{156}$$

is also the transverse Mercator projection.

An alternative approach to developing a transverse Mercator projection is to expand equation (148) as a power series.

Following Lauf (1983), consider a point *P* having isometric coordinates ψ,ω linked to an approximate location ψ_0,ω_0 by very small corrections $\delta\psi,\delta\omega$ such that $\psi=\psi_0+\delta\psi$ and $\omega=\omega_0+\delta\omega$; equation (148) becomes

$$\begin{aligned} Y+iX &= f\left(\psi+i\omega\right) \\ &= f\left\{\left(\psi_0+\delta\psi\right)+i\left(\omega_0+\delta\omega\right)\right\} \\ &= f\left\{\left(\psi_0+i\omega_0\right)+\left(\delta\psi+i\delta\omega\right)\right\} \\ &= f\left(z_0+\delta z\right)=f\left(z\right) \end{aligned} \tag{157}$$

The complex function $f(z)$ can be expanded by a Taylor series [see equation (139)]

$$\begin{aligned} f\left(z\right) &= f\left(z_0\right)+\delta z\, f^{(1)}\left(z_0\right)+\frac{\left(\delta z\right)^2}{2!}f^{(2)}\left(z_0\right) \\ &\quad+\frac{\left(\delta z\right)^3}{3!}f^{(3)}\left(z_0\right)+\cdots \end{aligned} \tag{158}$$

where $f^{(1)}(z_0),\, f^{(2)}(z_0)$, etc. are first, second and higher order derivatives of the function $f(z)$ can be evaluated at $z=z_0$. Choosing, as an approximate location, a point on the central meridian having the same isometric latitude as *P,* then $\delta\psi=0$ (since $\psi=\psi_0+\delta\psi$ and $\psi_0=\psi$) and $\delta\omega=\omega$ (since $\omega=\omega_0+\delta\omega$ and $\omega_0=0$), hence $z_0=\psi_0+i\omega_0=\psi$ and $\delta z=\delta\psi+i\delta\omega=i\omega$.

The complex function $f(z)$ can then be written as

$$\begin{aligned} f\left(z\right) &= f\left(\psi+i\omega\right) \\ &= f\left(\psi\right)+i\omega f^{(1)}\left(\psi\right)+\frac{\left(i\omega\right)^2}{2!}f^{(2)}\left(\psi\right) \\ &\quad+\frac{\left(i\omega\right)^3}{3!}f^{(3)}\left(\psi\right)+\cdots \end{aligned} \tag{159}$$

$f^{(1)}(\psi)$, $f^{(2)}(\psi)$, etc. are first, second and higher order derivatives of the function $f(\psi)$.

Substituting equation (159) into equation (157) and equating real and imaginary parts (noting that $i^2=-1,\ i^3=-i,\ i^4=1$, etc. and $f(\psi)=M$) gives

$$\begin{aligned} X &= \omega\frac{dM}{d\psi}-\frac{\omega^3}{3!}\frac{d^3M}{d\psi^3}+\frac{\omega^5}{5!}\frac{d^5M}{d\psi^5}-\frac{\omega^7}{7!}\frac{d^7M}{d\psi^7}+\cdots \\ Y &= M-\frac{\omega^2}{2!}\frac{d^2M}{d\psi^2}+\frac{\omega^4}{4!}\frac{d^4M}{d\psi^4}-\frac{\omega^6}{6!}\frac{d^6M}{d\psi^6}+\cdots \end{aligned} \tag{160}$$

In this alternative approach, the transformation from the plane to the ellipsoid is represented by the complex expression

$$\psi+i\omega=F\left(Y+iX\right) \tag{161}$$

And similarly to before, the complex function $F(Y+iX)$ can be expanded as a power series giving

$$\begin{aligned} \psi+i\omega &= F\left(Y\right)+iXF^{(1)}\left(Y\right)+\frac{\left(iX\right)^2}{2!}F^{(2)}\left(Y\right) \\ &\quad+\frac{\left(iX\right)^3}{3!}F^{(3)}\left(Y\right)+\cdots \end{aligned} \tag{162}$$

When $X=0$, $\omega=0$; but when $X=0$ the point $P(\phi,\omega)$ becomes $P_1(\phi_1,0)$, a point on the central meridian having latitude ϕ_1 known as the foot-point latitude. Now ψ_1 is the isometric latitude for the foot-point latitude and we have $F(Y)=\psi_1$

Substituting equation (162) into equation (161) and equating real and imaginary parts gives

$$\begin{aligned} \psi &= \psi_1-\frac{X^2}{2!}\frac{d^2\psi_1}{dY^2}+\frac{X^4}{4!}\frac{d^4\psi_1}{dY^4}-\frac{X^6}{6!}\frac{d^6\psi_1}{dY^6}+\cdots \\ \omega &= X\frac{d\psi_1}{dY}-\frac{X^3}{3!}\frac{d^3\psi_1}{dY^4}+\frac{X^5}{5!}\frac{d^5\psi_1}{dY^5}-\frac{X^7}{7!}\frac{d^7\psi_1}{dY^7}+\cdots \end{aligned} \tag{163}$$

The first of equations (163) gives ψ in terms of ψ_1 but we require ϕ in terms of ϕ_1. Write the first of equations (163) as

$$\psi=\psi_1+\delta\psi \tag{164}$$

where

$$\delta\psi=-\frac{X^2}{2!}\frac{d^2\psi_1}{dY^2}+\frac{X^4}{4!}\frac{d^4\psi_1}{dY^4}-\frac{X^6}{6!}\frac{d^6\psi_1}{dY^6}+\cdots \tag{165}$$

And latitude $\phi = g(\psi) = g(\psi_1 + \delta\psi)$ can be expanded as another power series

$$\phi = g(\psi_1) + \delta\psi\, g^{(1)}(\psi_1) + \frac{(\delta\psi)^2}{2!} g^{(2)}(\psi_1) + \frac{(\delta\psi)^3}{3!} g^{(3)}(\psi_1) + \cdots \tag{166}$$

Noting that $g(\psi_1) = \phi_1$ we may write the transformation as

$$\begin{aligned} \phi &= \phi_1 + \delta\psi \frac{d\phi_1}{d\psi_1} + \frac{(\delta\psi)^2}{2!}\frac{d^2\phi_1}{d\psi_1^2} + \frac{(\delta\psi)^4}{4!}\frac{d^4\phi_1}{d\psi_1^4} + \cdots \\ \omega &= X\frac{d\psi_1}{dY} - \frac{X^3}{3!}\frac{d^3\psi_1}{dY^4} + \frac{X^5}{5!}\frac{d^5\psi_1}{dY^5} - \frac{X^7}{7!}\frac{d^7\psi_1}{dY^7} + \cdots \end{aligned} \tag{167}$$

REFERENCES

Adams, O.S., (1921), *Latitude Developments Connected with Geodesy and Cartography,* Special Publication No. 67, U.S. Coast and Geodetic Survey, Department of Commerce, Washington.

Bromwich, T.J.I'A., (1991), *An Introduction to the Theory of Infinite Series,* 3rd edition, Chelsea Publishing Company, New York.

Bugayevskiy, L.M. and Snyder, J.P., (1995), *Map Projections: A Reference Manual,* Taylor & Francis, London.

Deakin, R.E. and Hunter, M.N., (2010a), *Geometric Geodesy Part A,* School of Mathematical and Geospatial Sciences, RMIT University, Melbourne, Australia, January 2010.

Deakin, R.E. and Hunter, M.N., (2010b), *Geometric Geodesy Part B,* School of Mathematical and Geospatial Sciences, RMIT University, Melbourne, Australia, January 2010.

Engsager, K. and Poder, K., (2007), 'A highly accurate world wide algorithm for the transverse Mercator mapping (almost)', Proceedings of XXIII International Cartographic Conference (ICC2007), 4-10 August, Moscow.

GDAV2.3 (2009), *Geocentric Datum of Australia - Technical Manual,* Version 2.3, Amendment 1, Intergovernmental Committee on Surveying and Mapping, July 2009. URL `http://www.icsm.gov.au/gda/gdatm/index.html`

GeoTrans, (2010), The Geographic Translator, Edition 3, National Geospatial-Intelligence Agency, URL `http://earth-info.nima.mil/GandG/geotrans`

Helmert, F.R., (1880), *Die mathematischen und physikalischen Theorieen der höheren Geodäsie,* Vol. 1, *Die mathematischen Theorieen,* Leipzig.

Karney, C.F.F., (2010), 'Transverse Mercator projection with an accuracy of a few nanometres', *Journal of Geodesy* (in press).

Krüger, L., (1912), *Konforme Abbildung des Erdellipsoids in der Ebene,* New Series 52, Royal Prussian Geodetic Institute, Potsdam. DOI 10.2312/GFZ.b103-krueger28

Lambert, J.H., (1772), *Beyträge zum Gebrauche der Mathematik und deren Anwendung III,* Part VI: Anmerkungen und Zusätze zur Entwerfung der Land- und Himmelscharten, pp.105-199, URL `http://books.google.com/books?id=sf82AAAAMAAJ`

Lauf, G.B., (1983), *Geodesy and Map Projections,* Tafe Publications, Collingwood, Australia.

Lee, L.P., (1976), *Conformal projections based on Jacobian elliptic functions,* Cartographica, Monograph 16, supplement 1 to Canadian Cartographer, Vol. 13, 128 pages.

Redfearn, J.C.B., (1948), 'Transverse Mercator formulae', *Empire Survey Review,* Vol. 9, No. 69, pp. 318-322, July 1948

Schreiber, O., (1866), *Theorie der Projektionsmethode der hannoverschen Landsvermessung,* Hannover: Hahnsche Hofbuchhandlung.

Schreiber, O., (1897), *Die konforme Doppelprojektion der trigonometrischen Abteilung der Königl,* Preussichen Landesaufnahme, Berlin.

Snyder, J.P., (1987), *Map Projections–A Working Manual,* Professional Paper 1395, U.S. Geological Survey, Government Printing Office , Washington.

Snyder, J.P., (1993), *Flattening The Earth - Two Thousand Years of Map Projections,* The University of Chicago Press, Chicago.

SOED, (1993), *The New Shorter Oxford English Dictionary,* Ed. Leslie Brown, Clarendon Press, Oxford.

Sokolnikoff, I.S. and Redheffer, R.M., (1966), *Mathematics of Physics and Modern Engineering,* 2nd edition, International Student Edition, McGraw-Hill Kogakusha, Ltd. Tokyo.

Thomas, P.D., (1952), *Conformal Projections in Geodesy and Cartography,* Special Publication No. 251, U.S. Coast and Geodetic Survey, Department of Commerce, Washington. URL `http://docs.lib.noaa.gov/rescue/cgs_specpubs/QB27U35no2511952.pdf`

Yang, Q., Snyder, J.P. and Tobler, W., (2000), *Map Projection Transformation,* Taylor & Francis, London.

NOTE ON MINOR CORRECTIONS TO ORIGINAL VERSION

Equation (46) in the original version had the coefficient $D_6 = \frac{151}{48}n^3 - \cdots$. This was incorrect and has been amended to $D_6 = \frac{151}{96}n^3 - \cdots$ in this version. This error was discovered by John Nolton, retired geodetic surveyor, and kindly reported to the authors. Thank you, John. (Rod Deakin, randm.deakin@gmail.com)